EXAMPLES

VOL. I.

T. C. HANSARD, Paternoster-row Press.

T. C. HANSARD, Paternoster-row Press.

TRANSLATOR'S

PREFACE.

DURING the period of my Undergraduateship at Cambridge, I felt, in common with the greater part of my fellow-students, the want of a copious and varied collection of examples to the various subjects treated in the elementary parts of Algebra. The work which I have now translated from the German of Meyer Hirsch, supplies the deficiency I then so strongly felt. It is not only far more extensive than some of those attempts which have been recently made, but arranged with much taste, and with suitable regard to the natural progress of the student; and, with the exception of equations of the higher orders, embraces all those topics which are usually considered as proper to be treated of by Algebra, such as the Indeterminate Analysis, the Diaphantine Analysis, Annuities, Chances, &c.

The author, MEYER HIRSCH, is not only one of the ablest of the continental mathematicians, but one of the most successful teachers of our time. Of the value of his book no higher testimony needs, or can be given, than the great number of editions the original has run through in the course of a few years.

Westwell Vicarage, Kent,
Jan. 1827.

HIRSCH'S

COLLECTION OF

EXAMPLES, FORMULÆ, & CALCULATIONS,

ON THE

LITERAL CALCULUS

AND

ALGEBRA.

TRANSLATED FROM THE GERMAN,

BY THE

REV. J. A. ROSS, A.M.

TRANSLATOR OF HIRSCH'S INTEGRAL TABLES.

VOL. II.

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T. C. HANSARD, Paternoster-Row Press.

PREFACE.

THAT I have discovered the general solution of equations, my readers may very probably know already, from the announcement which I have ordered to be put in the public papers, for such a discovery deserves the utmost publicity. A complete history of the unsuccessful endeavours of my predecessors for the same purpose, would appear like a panegyric on myself, and therefore I shall only relate what exactly belongs to the matter.

In the tenth part of the "*Memorie di Matematica e di Fisica della Societa Italiana della Scientia*," p. 1 (1803), a celebrated Analyst, M. Ruffini, gave a proof of the impossibility of such a solution : no very elaborate discussions are required to show its insufficiency. Read his proof, and compare it with my solution, and it will be found that M. Ruffini, in recounting the possible cases, has never thought of this mode of solution. M. Ruffini tries to depress the equation for the assumed function, by taking as many equal formulæ as are proper for his purpose. I do just the contrary : with me the assumed functions, with the

exception of a single condition, are always arbitrary: all their forms may be different, and the depression will be produced by making them dependent upon resolvable equations, the coefficients of which depend on other resolvable equations, the coefficients of which depend again on other resolvable equations, &c. As, for instance, with me the assumed function for the equation of the fifth degree originally depends on an equation of the 120th degree: this I reduce, first to a double equation of the fifth degree; its coefficient, which is still depending on an equation of the 24th degree, I make depending on an equation of the fourth degree, the coefficients of which only still depend on equations of the sixth degree. I reduce these equations again to equations of the third degree, the coefficients of which, lastly, depend on equations of the second degree. Of all this process, there is, with the exception of the reduction to the double equation, not the slightest indication in the proof of M. Ruffini. Moreover, this Analyst has only shown that none of the methods he was acquainted with could succeed, and in this respect his proof is, no doubt, very masterly. His error can be no reflection on his well-deserved reputation, for he has shown to his successors the paths they must avoid, and has thus put them on the right course of investigation.

M. Lagrange gave, in the third volume of the new Memoirs of the Berlin Academy of Sciences, an incomparable analysis of the methods of Tschirnhausen, Euler, and Bezout, which I have adopted in the sixth chapter,

with a few alterations suitable to my purpose. He showed, that when μ is a prime number,* all these methods lead at last to a reduced equation, the coefficients of which are those functions of the roots x', x'', x'''......$x^{(\mu)}$, which change when you change only the $\mu-2$ last roots among each other, but leave both the first in their places, and that, therefore, the coefficients depend on equations of the $1 . 2 . 3 ...... \mu$ second degree; and consequently an equation of the fifth degree, on an equation of the sixth degree. For the explanation of his method, he takes the equation of the fifth degree as an example, and shows how to begin to form the reduced equation. He denotes the roots of the reduced equations by z', z'', z''', z^{IV}, z^{V}, z^{VI}, and finds the value of them in* x', x'', x''', x^{IV}, x^{V}. From this he calculates the first coefficients, and says that you can find the other coefficients in a similar way. He concludes by saying "but we will not enter into the execution of this calculation, which, with all its immense labour, would not afford any clue to the resolution of equations of the fifth degree; for as the reduced equation for z is of the sixth degree, it is not resolvable, unless it is to be brought to a lower degree than the fifth. But this seems to me to be almost impossible, according to the form of the roots z', z'', &c."

But from these very forms, I affirm that the solution of the reduced equation is possible. For the functions z'', z'''

* μ is with him, what n is with me.

* Pages 432 and 433 of Euler's Introduction, translated by Michelsen.

z^{IV}, z^{V}, z^{VI} are derived from z, as M. Lagrange observed himself, when you change the roots x''', x^{IV}, x^{V}, among each other. According to my notation, therefore, the roots of the reduced equation admit of being presented by $f:(12345)$, $f:(12453)$, $f:(12534)$, $f:(12435)$, $f:(12354)$, and $f:(12543)$, and with these notations the functions z', z^{IV}, z^{VI}, z''' z'', z^{V}, correspond according to the order in which they are here put. The three former formulæ $f:(12345)$, $f:(12453)$, $f:(12534)$, form evidently a cyclical period of the three last roots, as well as the formulæ $f:(12435)$, $f:(12354)$, $f:(12543)$. If we therefore combine the three functions z', z^{IV}, z^{VI}, in one equation of the third degree, the coefficients of them can only have besides their value another one, namely, that which the change of x''' with x^{IV} gives. These coefficients, therefore, have no more than two unequal values, and consequently they depend only on equations of the second degree, or, what is the same, the reduced equation of the sixth degree can be divided into two equations of the third degree, the one of which has the roots z', z^{IV}, z^{VI}, the other the roots z''', z'', z^{V}. That this simple observation escaped the keen penetration of a Lagrange, looks indeed like a miracle. I am not the inventor, it is he; but he did not know it. Whether I should have found the solution without him, may be doubted.

I come now to my method of solution: it is very simple, uniform for all degrees, and as general as could be desired. It gives, not one solution, but as many as we

please; for the functions I mark with ϕ are quite arbitrary. However, the actual calculation is very troublesome, and even in the sixth degree is scarcely practicable without resorting to particular artifices. We cannot escape the difficulties of calculation, when the degree of an equation is a prime number. When, however, it is a compound number, we have, no doubt, a method, which leads more rapidly to the result: it will be reserved for the third volume, in which I shall give also the solution of the equations of the 5th, 6th, and 7th degree. The Combinatorial Analysis is here of great service; and with its help I shall perhaps be able at once to exhibit the reduced equation with little more trouble than the mere combinatorial operations.

A brief sketch of the contents of this volume will not be here improper. I begin with the Symmetrical Functions; they are the foundation of all others. The two first chapters treat of them; the first gives the recurred solution, the second the independent one. Generality was the object I aimed at. The third chapter treats of the Non-symmetrical Functions. They are derived altogether from certain equations, which I call transformed equations. It is shown how to find the equal formulæ of these functions, when their nature is given by certain properties; and how to form a transformed equation of the unequal formulæ. The numberless references to it require particular observation. The utility of some of these propositions will appear in the sequel. The fourth chapter treats of Elimination. I was not obliged to be

so minute as Bezout in his *Theorie générale des Equations Algebraiques*, who confines himself exclusively to this subject. My work would have become too voluminous. Should my readers wish for further details, this may be done in an appendix. The fifth chapter treats of the properties of the roots of the equation $x^n - 1 = 0$. Waring and Euler were my conductors. The labours of Lagrange, in the Memoirs of the Berlin Academy, gave me the materials for the sixth chapter. I desire my readers to bestow particular attention on the seventh chapter; its value will be shown in the third volume of this work. The eighth chapter treats of the General Solution of equations, but must be regarded only as a sketch.

My reader is no longer the same whom I thought of in the Collection, the continuation of which I now give him: he has gone much further in the sciences. The Combinatorial Analysis is no longer strange to him: he has also made already considerable progress in the Differential Calculus. Provided with this knowledge, he will, I trust, find my book not entirely useless. He will not remain where I have remained: he will look further. I do not lead him through an unfertile, but, for want of labour, an uncultivated field. For since the Differential and Integral Calculation employed the Analysts, Algebra has been little thought of.

The next part will contain, besides the deeper researches about the general solution of equations, a great many

other subjects, and amongst them the important, almost inexhaustible, one, of the Analysis of Equations. I shall constantly, as far as my leisure hours permit, labour on the Third Part to hasten its appearance as much as possible. But if on all these subjects the same pains be bestowed, some time must elapse before its appearance. However, not to let my readers wait for what belongs to equations in particular too long, I am inclined to prepare for the next *fair* a *supplement* of about four or five sheets on this subject, and in it to communicate the complete solution of the general equations of the fifth, sixth, and seventh degree.

Berlin, October, 1808.

TRANSLATOR'S

PREFACE.

SINCE the publication of Waring's *Meditationes Algebraicæ* there has not appeared, either in this country or on the continent, so elaborate and able a treatise on the theory of Equations as that of MEYER HIRSCH. Its merit has long been recognized, and the work held in the highest esteem by those who were able to read it; a small number, undoubtedly in this country, where the German language is so rarely understood by those who devote themselves to these studies.

As a treatise on equations of the higher orders, it is not less admirable for the clear and simple manner in which the theories are laid down, than for the numerous and apt examples by which they are elucidated.

It is necessary to remind the reader, however, that the Solution of the General Equation on the imagined discovery of which Meyer Hirsch so warmly congratulates himself in his preface, turns out, on examination, like all the other attempts of the same kind, to be a failure. What he has written on the subject, I have, nevertheless, permitted to remain; because, in the first place, these speculations occupy only a small space; next, because they are highly curious and interesting; and finally,

because I did not think myself at liberty to mutilate a work which I undertook only to translate.

J. A. ROSS.

Westwell-Vicarage, Kent,
January, 1827.

Extract from Meyer Hirsch's Preface to his Integral Tables.

"At the end of this preface I must observe, that I was mistaken when I maintained, in my collection of Problems on the Theory of Equations, that the general solution of equations was not only practicable, but even thought I had found it. The eighth chapter of the above-mentioned work must, therefore, be read with mistrust. It is true that I have found the solution of a number of very remarkable equations, that do not admit of being analysed, but by no means their general solution, in the sense in which Euler, Lagrange, and other great Analysts take these words (general solution); for I am now convinced of the impossibility of effecting it. The mistake arose from haste, and is so readily discovered, that every person who reads so far, will easily perceive it."

I.—On the roots of equations, the sums of their powers, and the products of these powers, and symmetrical functions in general.

SECTION I.

In all good elementary books on Algebra it is shewn, that the first part of an equation of the undetermined nth degree......(ψ)......

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + Px + Q = o$$

may always be considered as a product of n simple factors of the form $x-a$, $x-b$, $x-c$, $x-d$, &c., and that then a, b, c, d, &c. are all the values of the unknown magnitude x, which verify the equation (ψ). If we actually multiply these factors, and compare the product resulting from this multiplication with the polynomial in the first part of the equation, then we have the following relations between the coefficients A, B, C, D, &c. of the equation, and its roots a, b, c, d, &c.:

$$-A = a + b + c + d + \&c.$$
$$B = ab + ac + ad + bc + bd + cd + \&c.$$
$$-C = abc + abd + acd + bcd + \&c.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\pm Q = abcd \ \&c.$$

Thus the first coefficient A, with the sign changed, is always the sum of all the roots; the second coefficient B, with its own sign, is the sum of all the products of every two of these roots; the third coefficient C, with the sign changed, is the sum of all the products of every three of these roots; and generally the undetermined mth coefficient, with its sign changed or not, according as m is an odd or even number, is the sum of all the products which arise from the combination of all the roots, taken m and m together; finally, the last term Q, (which may also be considered as the coefficient of x^0,) with its sign changed or not, according as n is an odd or even number, is merely the product of all the roots.

The coefficients A, B, C, D, &c., are consequently no other than the aggregates of the combinations of the roots a, b, c, d, &c., taken singly (one by one), two and two, three and three, four and four, &c., or, to express myself more precisely according to Hindenburg, the aggregates of the combinations without repetitions of the first, second, third, &c. class. How these combinations may be easily represented, will be shown under the head of combinations, which are here supposed to be known only in their first principles.

SECTION II.

In the following pages, certain notations are frequently used, which, in fact, are already known to the greater part of my readers; the meaning of these notations, however, in order to prevent mistakes and confusion, I shall give in this place.

1. When determinate or indeterminate magnitudes are spoken of, all algebraical expressions, in which these two kinds of magnitudes are in any way involved, are called functions. We then use the formula: "This or that expression is a function of these or those magnitudes"—because we only mention the indeterminate magnitudes, omitting the determinate.

On account of the particular use which we shall make of functions in this Work, I wish it to be remembered, once for all, that here (when the contrary is not expressly mentioned), only such functions are meant, as may immediately be determined by means of the six arithmetical operations of Addition, Subtraction, Multiplication, Division, Involution and Evolution, so soon as the magnitudes contained in these functions are known, and when they do not contain a magnitude considered as indeterminate either as the exponent of powers, or the index of roots.

2. A rational function is one in which either there are no irrational magnitudes, or one in which at least those magnitudes which are considered as indeterminate are not under the radical sign; in the contrary case the function is an irrational one.

3. An integral function is one in which either there is no denominator, or in which, at least, those magnitudes which are considered as indeterminate, are not found in the denominator; in the contrary case it is called a fractional function.

The coefficients A, B, C, &c. of the equation (ψ) in

§ I., are consequently integral and rational functions of the roots a, b, c, d, &c., so long as these magnitudes are considered as indeterminate. Here no reference is made to the particular properties of the magnitudes a, b, c, d, &c. themselves; consequently these may be rational or irrational, integral or fractional, and, generally, may have every possible form, or they may even be functions of other magnitudes.

4. Those functions which are here called symmetrical, are those in which the indeterminate magnitudes are so combined, that, independently of the particular values of these magnitudes, no change takes place in the value of the function, however these magnitudes are substituted for one another.

The coefficients A, B, C, D, &c. of the equation (ψ) in § I., are $\therefore$ symmetrical functions of the roots a, b, c, d, &c.; thus they remain the same when a is substituted for b, or b for c, or a for c, and b for d, and so in like manner of other substitutions.

From this definition it immediately follows, that the sums, remainders, products, quotients, powers and roots of symmetrical functions, are again symmetrical functions, provided the functions, which are combined together by addition, subtraction, multiplication and division, contain all the same indeterminate magnitudes, and in the same number. Thus the expression

$$\frac{(ab + ac + bc)^m + abc}{\sqrt[n]{(abc - a - b - c)}}$$

is a symmetrical function of a, b, c, because $ab+ac+bc$,

abc, $a+b+c$, are functions of this kind. Generally, every function of one or more symmetrical functions is always again a symmetrical function, when these last contain the same indeterminate magnitudes and in the same number.

Now it may be shown that every rational, integral, or fractional function of the roots of an equation, however constituted, may always be expressed rationally by the coefficients of this equation. This highly important relation between the coefficients and the roots, has thrown more light on the theory of equations than any other; and should human genius ever succeed in fully discovering the secret of its solution, so far as this is possible, it will probably be by such inquiries as are exactly founded on this very property.

SECTION III.

For the sake of brevity and perspicuity, I shall use the following symbols:

Let the sum of all the roots of an equation, their squares, their cubes, their biquadrates, and, in general, the sum of their μth powers, be represented by (1), (2), (3), (4) . . . . . (μ), so that only the exponents, but not the roots, are indicated, because the latter are not considered in the present case. Therefore,

$$(1) = a + b + c + d + e + \&c.$$
$$(2) = a^2 + b^2 + c^2 + d^2 + e^2 + \&c.$$
$$(3) = a^3 + b^3 + c^3 + d^3 + e^3 + \&c.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\mu) = a^\mu + b^\mu + c^\mu + d^\mu + e^\mu + \&c.$$

If the roots a, b, c, d, &c. taken two and two, be combined in all possible ways, and in each such combina-

tion every alternate root raised to the power α, the other to the power β, then the sum of all the products thus obtained may be represented by $(\alpha\beta)$. Consequently, when only four roots are assumed, the equation (ψ), in § I., is of the fourth degree,

$$(\alpha\beta) =$$
$$a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + a^\alpha d^\beta + a^\beta d^\alpha +$$
$$b^\alpha c^\beta + b^\beta c^\alpha + b^\alpha d^\beta + b^\beta d^\alpha + c^\alpha d^\beta + c^\beta d^\alpha.$$

In a similar manner $(\alpha\beta\gamma)$ denotes the sum of all the products which arise from the combination of all the roots, taken three and three, and in each such combination raising one root to the power α, another to the power β, and the third to the power γ, and this in as many ways as possible. When again only four roots are assumed, we $\therefore$ have

$$(\alpha\beta\gamma) =$$
$$a^\alpha b^\beta c^\gamma + a^\alpha b^\gamma c^\beta + a^\beta b^\alpha c^\gamma + a^\beta b^\gamma c^\alpha + a^\gamma b^\alpha c^\beta + a^\gamma b^\beta c^\alpha +$$
$$a^\alpha b^\beta d^\gamma + a^\alpha b^\gamma d^\beta + a^\beta b^\alpha d^\gamma + a^\beta b^\gamma d^\alpha + a^\gamma b^\alpha d^\beta + a^\gamma b^\beta d^\alpha +$$
$$a^\alpha c^\beta d^\gamma + a^\alpha c^\gamma d^\beta + a^\beta c^\alpha d^\gamma + a^\beta c^\gamma d^\alpha + a^\gamma c^\alpha d^\beta + a^\gamma c^\beta d^\alpha +$$
$$b^\alpha c^\beta d^\gamma + b^\alpha c^\gamma d^\beta + b^\beta c^\alpha d^\gamma + b^\beta c^\gamma d^\alpha + b^\gamma c^\alpha d^\beta + b^\gamma c^\beta d^\alpha.$$

In general, $(\alpha\beta\gamma\delta\ldots\ldots\kappa)$, when m is the number of the letters α, β, γ, δ, ... κ, denotes a sum of the products which arise from the combination of all the roots to the mth class, and in each class one root is raised to the power α, another to the power β, a third to the power γ, and so on, in as many ways as possible.

In order $\therefore$ to represent actually the expression $(\alpha\beta\gamma\delta\ldots\ldots\kappa)$, find all the combinations of the roots a, b, c, d, &c. taken m and m together, give the roots in each such combination the exponents α, β, γ, ..., κ, and then

permute the exponents in all possible ways. By this method we get

$$[\alpha\beta\gamma\delta] =$$
$$a^\alpha b^\beta c^\gamma d^\delta + a^\alpha b^\beta c^\delta d^\gamma + \ldots\ldots + a^\delta b^\gamma c^\beta d^\alpha +$$
$$a^\alpha b^\beta c^\gamma e^\delta + a^\alpha b^\beta c^\delta e^\gamma + \ldots\ldots + a^\delta b^\gamma c^\beta e^\alpha +$$
&c.

$$[\alpha\alpha\alpha\beta\beta] =$$
$$a^\alpha b^\alpha c^\alpha d^\beta e^\beta + a^\alpha b^\alpha c^\beta d^\alpha e^\beta + \ldots + a^\beta b^\beta c^\alpha d^\alpha e^\alpha +$$
$$a^\alpha b^\alpha c^\alpha d^\beta f^\beta + a^\alpha b^\alpha c^\beta d^\alpha f^\beta + \ldots + a^\beta b^\beta c^\alpha d^\alpha f^\alpha +$$
&c.

In order to render this notation more convenient, it would be better to use the repeating exponents in those terms where an exponent occurs more than once; thus, for instance, $[\alpha^3\beta^2]$ instead of $[\alpha\alpha\alpha\beta\beta]$, and $[\alpha^2\beta^2\gamma^2]$ instead of $[\alpha\alpha\beta\beta\gamma\gamma]$.

By § I., the relation between the coefficients and the roots of an equation may be thus represented:

$-A = [1]$, $B = [11] = [1^2]$, $-C = [111] = [1^3]$, $D = [1111]$, $= [1^4]$, $-E = [11111]$, $= [1^5]$, &c.

From the construction of the function $[\alpha\beta\gamma\delta\ldots\ldots\kappa]$, it follows, that it belongs to symmetrical functions, because it undergoes no change when the roots a, b, c, d, &c. are substituted for one another.

The function $[\alpha\beta\gamma\delta\ldots\kappa]$, or more generally $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}]$, I also sometimes call a numerical expression. The exponents α, β, γ, δ, ... κ, are called radical exponents, in order to distinguish them from the repeating exponents a, b, c, d, ... k.

The radical exponents may also be negative, and then

$$[-1] = a^{-1} + b^{-1} + c^{-1} + \&c. = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \&c.$$

$$[-2] = a^{-2} + b^{-2} + c^{-2} + \&c. = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \&c.$$

. . . .

$$[-\mu] = a^{-\mu} + b^{-\mu} + c^{-\mu} + \&c. = \frac{1}{a^\mu} + \frac{1}{b^\mu} + \frac{1}{c^\mu} + \&c.$$

and in the same manner

$$[\overline{-\alpha\beta}] = a^{-\alpha}b^\beta + a^\beta b^{-\alpha} + a^{-\alpha}c^\beta + a^\beta c^{-\alpha} + \&c.$$

$$= \frac{b^\beta}{a^\alpha} + \frac{a^\beta}{b^\alpha} + \frac{c^\beta}{a^\alpha} + \frac{a^\beta}{c^\alpha} + \&c.$$

$$[\overline{-\alpha\overline{-\beta}}] = a^{-\alpha}b^{-\beta} + a^{-\beta}b^{-\alpha} + a^{-\alpha}c^{-\beta} + a^{-\beta}c^{-\alpha} + \&c.$$

$$= \frac{1}{a^\alpha b^\beta} + \frac{1}{a^\beta b^\alpha} + \frac{1}{a^\alpha c^\beta} + \frac{1}{a^\beta c^\alpha} + \&c.$$

and in a similar way it obtains with the numerical expressions, in which more than one negative radical exponent occur.

SECTION IV.

Prob. To find the number of terms of which the numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ consists.

Solution 1. Let the number of the roots a, b, c, d, &c. $= n$, and the number of the radical exponents α, β, γ, δ, $\ldots \kappa = m$.

2. The terms which compose the numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ may be found by combining the n roots a, b, c, d, &c. to the mth class, and by permuting in each such combination the m radical exponents α, β, γ, δ, $\ldots \kappa$ (§ III.) The number of these terms is consequently equal to the product of the number of combinations of n things to the mth class, and the number of permutations of m different things.

3. But the number of combinations of n things, taken m and m, is

$$= \frac{n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2 \,.\, n-m+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots m-1 \,.\, m}$$

and the number of permutations of m things is

$$= 1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots m-1 \,.\, m.$$

4. Hence the number of the terms of the numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$

$$= \frac{n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots m} \times 1 \,.\, 2 \,.\, 3 \ldots m$$

$$= n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2 \,.\, n-m+1$$

SECTION V.

PROB. To find the number of the terms of the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots\ldots \kappa^{\mathfrak{k}}]$.

Solution 1. Let the number of all the roots a, b, c, d, &c. $= n$; let the number of the radical exponents, without reference to their equality or difference, or, which is the same in this case, let the sum of the repeating exponents $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+ \ldots\ldots +\mathfrak{k}$, &c. $= m$.

2. Since each term of $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots\ldots \kappa^{\mathfrak{k}}]$ contains m of the roots a, b, c, d, &c., in order to find the number of terms, we must here, as in the preceding section, multiply the number of combinations of n roots taken m and m by the number of permutations of their m radical exponents.

3. But the former

$$= \frac{n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2 \,.\, n-m+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots m-1 \,.\, m}.$$

The second will be obtained by finding how often the letters α, β, γ, δ, ..., κ, can be transposed in a series of elements $\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots\ldots \kappa^{\mathfrak{k}}$, in which one letter occurs $\mathfrak{a}$ times, another $\mathfrak{b}$, a third $\mathfrak{c}$, and so on. The number of these transpositions, however, is (as may be seen from the rule of combinations)

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots m-1 \cdot m}{1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots \times 1 \cdot 2 \ldots \mathfrak{k}}$$

4. If $\therefore$ these two numbers are multiplied together, we obtain the number of the terms of $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots\ldots \kappa^{\mathfrak{k}}]$

$$= \frac{n \cdot n-1 \cdot n-2 \ldots\ldots\ldots\ldots\ldots n-m+2 \cdot n-m+1}{1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 \cdot 2 \ldots \mathfrak{k}}$$

Corollary. If the m radical exponents are all different from one another, then, by the preceding section, the number of terms $= n \cdot n-1 \cdot n-2 \ldots n-m+2 \cdot n-m+1$. Hence it follows, that in the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots\ldots \kappa^{\mathfrak{k}}]$, the number of terms is less by $1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 \cdot 2 \ldots \mathfrak{k}$ than when the radical exponents are different.

Example. For the expression $[\alpha^4\beta^3\gamma^2]$, when the equation to which it relates is of the twelfth degree, we have $n = 12$, $\mathfrak{a} = 4$, $\mathfrak{b} = 3$, $\mathfrak{c} = 2$; $\therefore m = 9$. The number of terms, of which this expression consists, is consequently

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \times 1 \cdot 2 \cdot 3 \times 1 \cdot 2} = 277200$$

SECTION VI.

PROB. Let Σ denote the sum of all the combinations without repetitions of n letters a, b, c, d, &c. of the mth class; further, let Σ' be the sum of all the combinations of this class, which do not contain a; Σ'' the sum of all those which do not contain b; Σ''' the sum of all those which do not contain c, and so on; find the relation between $\Sigma' + \Sigma'' + \Sigma''' + \&c.$ and Σ.

Solution 1. If in Σ', Σ'', Σ''', &c. all the combinations are completed, or if $\Sigma = \Sigma' = \Sigma'' = \Sigma''' = \&c.$ then $\Sigma' + \Sigma'' + \Sigma''' + \&c. = n\,\Sigma$. But since in these some of the combinations are wanting, consequently their sum must be less than $n\,\Sigma$.

2. But it is evident, that each distinct combination contained in Σ must be wanting in the sum $\Sigma' + \Sigma'' + \Sigma''' + \&c.$ exactly as many times as it contains elements. Then supposing the different combinations in Σ consist of four letters, then, for instance, the combination $a\,b\,c\,d$ fails once in Σ', Σ'', Σ''', Σ^{IV}.

3. If $\therefore$, in general, m is the number of elements contained in each combination, then the sum of all the combinations which fail in $\Sigma' + \Sigma'' + \Sigma''' + \&c.$ is $m\,\Sigma$.

4. Hence and from 1. it follows, that

$$\Sigma' + \Sigma'' + \Sigma''' + \&c. = (n-m)\,\Sigma.$$

Corollary. Therefore $\Sigma' + \Sigma'' + \Sigma''' + \&c.$ for the first class $= (n-1)\,\Sigma$; for the second $= (n-2)\,\Sigma$; for the third $= (n-3)\,\Sigma$; and so on.

Example. Let Σ be the sum of all the combinations, taken three and three, of five elements a, b, c, d, e; then $\Sigma = abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$, $\Sigma' = bcd + bce + bde + cde$, $\Sigma'' = acd + ace + ade + cde$, $\Sigma''' = abd + abe + ade + bde$, $\Sigma^{iv} = abc + abe + ace + bce$, $\Sigma^{v} = abc + abd + acd + bcd$, and $\Sigma' + \Sigma'' + \Sigma''' + \Sigma^{iv} + \Sigma^{v} = 2\,\Sigma = (5 - 3)\ \Sigma$, as was required.

SECTION VII.

Prob. There are two equations,

$$x^n \ + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = o$$
$$x^{n-1} + A'x^{n-2} + B'x^{n-3} + C'x^{n-4} + \&c. = o$$

of which the second has the same number of roots as the first, except a: find the relation between the coefficients of these two equations.

Solution. The second equation may be obtained from the first, by dividing the latter by $x-a$. By actually performing this division, we obtain

$$x^{n-1} + (a + A)\, x^{n-1} + (a^2 + aA + B)\, x^{n-2} +$$
$$(a^3 + a^2A + aB + C)\, x^{n-3} + \&c. = o.$$

Hence now it follows, that $A' = a + A$, $B' = a^2 + aA + B$, $C' = a^3 + a^2A + aB + C$ &c. and, in general,

$$\overset{m}{A}{}' = a^{-m} + a^{m-1}A + a^{m-2}B + a^{m-3}C + \ldots\ldots + \overset{m}{A}$$

When $\overset{m}{\mathrm{A}}$ and $\overset{m}{\mathrm{A}}{}'$ denote the mth coefficients in the first and second equations.

SECTION VIII.

Prob. From a given equation to find the sums of the

squares, cubes, biquadrates, and, generally, the sum of any power of its roots, without knowing these roots, assuming that the exponent of this power is a whole positive number.

Solution 1. Let

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots\ldots + Px + Q = o$$

be the given equation, whose roots are a, b, c, d, &c. Further, let

$$x^{n-1} + A'x^{n-2} + B'x^{n-3} + C'x^{n-4} + \&c. = o$$
$$x^{n-1} + A''x^{n-2} + B''x^{n-3} + C''x^{n-4} + \&c. = o$$
$$x^{n-1} + A'''x^{n-2} + B'''x^{n-3} + C'''x^{n-4} + \&c. = o$$
&c.

be the n equations, which arise from dividing the given equation by $x-a$, $x-b$, $x-c$, &c. successively.

2. Then the coefficients A, B, C, D, &c. are the positive or negative sums of the letters, taken singly, two and two, three and three, four and four, and so on, of n roots a, b, c, d, &c.; the coefficients A', B', C', D', &c., the sums of the letters, taken singly, two and two, three and three, four and four, and so on, of the $n-1$ roots, b, c, d, e, &c.; the coefficients A'', B'', C'', D'', &c.; the positive or negative sums of the $n-1$ roots, a, c, d, e, &c. taken singly, two and two, three and three, four and four. Then (§ VI.)

$$A' + A'' + A''' + \&c. = (n-1)\ A$$
$$B' + B'' + B''' + \&c. = (n-2)\ B$$
$$C' + C'' + C''' + \&c. = (n-3)\ C$$
&c.

3. But from the preceding §

$$A' = a + A,\ A'' = b + A,\ A''' = c + A,\ \&c.$$

If we use the symbols in § III., we consequently have

$$A' + A'' + A''' + \&c. = (1) + nA$$

Since further (foregoing §)

$$B' = a^2 + aA + B,\ B'' = b^2 + bA + B,$$
$$B''' = c^2 + cA + B,\ \&c.$$

then we have

$$B' + B'' + B''' + \&c. = (2) + A\,(1) + nB$$

In like manner we find

$$C' + C'' + C''' + \&c. = (3) + A\,(2) + B\,(1) + nC$$
$$\&c.$$

4. From 2. and 3. we obtain the following equations:

$$(1) + nA = (n-1)\,A$$
$$(2) + A\,(1) + nB = (n-2)\,B$$
$$(3) + A\,(2) + B\,(1) + nC = (n-3)\,C$$
$$\&c.$$

or

$$(1) + A = o$$
$$(2) + A\,(1) + 2B = o$$
$$(3) + A\,(2) + B\,(1) + 3\,C = o$$

and in general

$$(m) + A\,(m-1) + B\,(m-2) + \ldots + \overset{m-1}{A}\,(1) + m\overset{m}{A} = o$$

where $\overset{m-1}{A}$, $\overset{m}{A}$, denote the $(m-1)$th and the mth coefficients when, $m < n$. But if $m =$ or $> n$, then the conclusions which have been drawn no longer obtain, because, in this case, the sixth section, on which they are founded, ceases to be applicable. We can, however, for this case find a similar equation by another method.

5. Thus, if we multiply the given equation by x^{m-n}, we obtain

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots\ldots + Px^{m-n+1} + Qx^{m-n} = 0$$

and if we substitute in this equation a, b, c, d successively for x, we have

$$a^m + Aa^{m-1} + Ba^{m-2} + \ldots\ldots + Pa^{m-n+1} + Qa^{m-n} = 0$$
$$b^m + Ab^{m-1} + Bb^{m-2} + \ldots\ldots + Pb^{m-n+1} + Qb^{m-n} = 0$$
$$c^m + Ac^{m-1} + Bc^{m-2} + \ldots\ldots + Pc^{m-n+1} + Qc^{m-n} = 0$$
&c.

If we add these equations together, we obtain

$$(m) + A\,(m-1) + B\,(m-2) + \ldots\ldots + P\,(m-n+1) + Q\,(m-n) = 0.$$

6. If in this equation we put $m = n$, then, because $(0) = a^0 + b^0 + c^0 + d^0 + \&c. = n$ we have

$$(n) + A\,(n-1) + B\,(n-2) + \ldots\ldots + P\,(1) + nQ = 0$$

7. By means of the equations found in 4. 5. and 6. we are now enabled to express the sum of every higher power by the sums of all the lower; and consequently, when these last are found, we are enabled to find the former. On account of the frequent use which is made of them in the following pages, I shall here arrange them together.

$$(1) + A = 0$$
$$(2) + A\,(1) + 2B = 0$$
$$(3) + A\,(2) + B\,(1) + 3C = 0$$
$$(4) + A\,(3) + B\,(2) + C\,(1) + 4D = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(n-1)+A(n-2)+B(n-3)+\ldots+M(1)+(n-1)P=0$$
$$(n)+A(n-1)+B(n-2)+\ldots+P(1)+nQ=0$$
$$(n+1)+A(n)+B(n-1)+\ldots+P(2)+Q(1)=0$$
$$(n+2)+A(n+1)+B(n)+\ldots+P(3)+Q(2)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(m)+A(m-1)+B(m-2)+\ldots\ldots+P(m-n+1)+Q(m-n)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

8. From these equations we successively obtain

$$(1) = -A$$
$$(2) = A^2 - 2B$$
$$(3) = -A^3 + 3AB - 3C$$
$$(4) = A^4 - 4A^2B + 2B^2 + 4AC - 4D$$
$$(5) = -A^5 + 5A^3B - 5AB^2 - 5A^2C + 5BC + 5Ad - 5E$$
$$(6) = A^6 - 6A^4B + 9A^2B^2 - 2B^3 + 6A^3C + 12ABC + 3C^2 - 6A^2D + 6BD + 6AE - 6F$$

&c.

and consequently the sum of the powers of the roots are expressed directly by the coefficients of the given equation.

Example. When the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, $A = -1$, $B = -19$, $C = 49$, $D = -30$. By substituting these values in the equations in 8. we obtain $(1) = 1$, $(2) = 39$, $(3) = -89$, $(4) = 723$, $(5) = -2849$, $(6) = 16419$, &c. Any person may easily convince himself of the truth of these results, by substituting in the first equation 1, 2, 3, -5 for x.

Remark. The formulæ in 8. are known by the name of the Newtonian Theorem, because Newton is supposed to be the first who has mentioned it. Other proofs of this theorem, and also much information relating to the subject itself, may be found amongst other matter in Kästner's Principles of Finite Analytical Magnitudes, third edition, p. 538, &c. also in Klügel's Mathematical Dictionary, part first, p. 465, &c. Art. Combination.

SECTION IX.

Prob. The sums of the powers of the roots of an equation, or the expressions (1), (2), (3), &c. are given: find the coefficients of this equation.

Solution. From the equations in 7. of the foregoing section, we obtain by transposition

$$A = -(1)$$

$$B = -\frac{A(1) + (2)}{2}$$

$$C = -\frac{B(1) + A(2) + (3)}{3}$$

$$D = -\frac{C(1) + B(2) + A(3) + (4)}{4}$$

&c.

By means of these equations we are enabled to determine successively the coefficients A, B, C, D, &c. when the numerical expressions (1), (2), (3), (4), &c. are given, as they are assumed to be in the problem.

SECTION X.

Prob. From a given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + Mx^3 + Nx^2 + Px + Q = 0$$

whose unknown roots are called a, b, c, d, &c. find another, whose roots are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c.

Solution. Substitute $\frac{1}{y}$ for x, and then multiply the whole equation by y^n; we then get

$$Qy^n + Py^{n-1} + Ny^{n-2} + My^{n-3} + \ldots\ldots + Ay + 1 = 0$$

or

$$y^n + \frac{P}{Q}y^{n-1} + \frac{N}{Q}y^{n-2} + \frac{M}{Q}y^{n-3} + \ldots\ldots + \frac{A}{Q} + \frac{1}{Q} = 0$$

and this is the required equation. For since $x = \frac{1}{y}$, then $y = \frac{1}{x}$, and since a, b, c, d, &c. are the values of x, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. are the values of y.

Corollary. The roots $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. in reference to the roots of the given equation, we term reciprocal roots. Therefore, if $x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Mx^3 + Nx^2 + Px + Q = 0$ be any equation, and $x^n + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \ldots\ldots + P'x + Q' = 0$, be the equation for its reciprocal roots; we then have

$$A' = \frac{P}{Q}, B' = \frac{N}{Q}, C' = \frac{M}{Q}, \&c.$$

SECTION XI.

Prob. Find the sum of a power of roots, when the exponent of this power is a whole negative number.

Solution. Let

$$x^n + Ax^{n-1} + \ldots\ldots + Mx^3 + Nx^2 + Px + Q = 0$$

be the given equation, and

$$x^n + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \ldots\ldots + P'x + Q' = 0$$

the equation for its reciprocal roots (foregoing section.) Then according to § VIII, when the numerical expressions (1), (2), (3), &c. are taken in reference to the second equation,

$$(1) + A' = 0$$
$$(2) + A'(1) + 2B' = 0$$
$$(3) + A'(2) + B'(1) + 3C' = 0$$

2. But (1), (2), (3), in reference to the second equation, are precisely what (-1), (-2), (-3), &c. are in reference to the first equation; we have ∴, when for A', B', C', &c. their values $\frac{P}{Q}$, $\frac{N}{Q}$, $\frac{M}{Q}$, &c., are substituted (foregoing section)

$$(-1) + \frac{P}{Q} = 0$$
$$(-2) + \frac{P}{Q}(-1) + \frac{2N}{Q} = 0$$
$$(-3) + \frac{P}{Q}(-2) + \frac{N}{Q}(-1) + \frac{3M}{Q} = 0$$

&c.

from which we can determine successively the sums of powers for negative exponents.

Example. When the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, we have $Q = -30$, $P = +49$, $N = -19$, $M = -1$: we have ∴ $(-1) = \frac{49}{30}$, $(-2) = \frac{1261}{900}$,

$(-3) = \frac{31159}{27000}$. Any one may readily be convinced of the accuracy of these results; for the roots of the given equation are 1, 2, 3, -5, consequently $(-1) = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \frac{49}{30}$, $(-2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} = \frac{1261}{900}$, $(-3) = 1 + \frac{1}{8} + \frac{1}{27} - \frac{1}{125} = \frac{31159}{27000}$, &c.

SECTION XII.

PROB. Express the symmetrical function $(\alpha\beta)$ by the sums of powers.

Solution. For the sake of greater perspicuity, I shall assume that there are only four roots, because this does not affect the general principle. Then

$$(\alpha) = a^\alpha + b^\alpha + c^\alpha + d^\alpha$$
$$(\beta) = a^\beta + b^\beta + c^\beta + d^\beta.$$

If we multiply these equations together, we then obtain

$$\begin{aligned}(\alpha)\,(\beta) = {} & a^{\alpha+\beta} + b^{\alpha+\beta} + c^{\alpha+\beta} + d^{\alpha+\beta} + \\ & a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + a^\alpha d^\beta + a^\beta d^\alpha + \\ & b^\alpha c^\beta + b^\beta c^\alpha + b^\alpha d^\beta + b^\beta d^\alpha + c^\alpha d^\beta + c^\beta d^\alpha.\end{aligned}$$

The first row of the second part of this equation $= (\alpha + \beta)$, and the remaining two rows $= (\alpha\beta)$; consequently we have

$$(\chi) \ldots\ldots (\alpha)\,(\beta) = (\alpha+\beta)\,(\alpha\beta)$$

and $\therefore$

$$(\alpha\beta) = (\alpha)\,(\beta) - (\alpha + \beta).$$

It is easily seen, that these conclusions obtain, let the number of the roots be what it may. Since then (α), (β),

$(\alpha + \beta)$, are only the sums of powers, what was required is now done.

The radical exponents α, β, may besides be either positive or negative. For example, if α be negative, then we have

$$(\overline{-\alpha}\beta) = (-\alpha)\,(\beta) = (\beta - \alpha)$$

and when α and β are both negative,

$$(\overline{-\alpha}\;\overline{-\beta}) = (-\alpha)\,(-\beta) = (-\alpha - \beta).$$

Corollary. Since we are always enabled to express the sums of powers, either for positive or negative exponents, by the coefficients of the given equation, in like manner we can always find the values of the expressions of the form $a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha +$ &c., $\dfrac{b^\beta}{a^\alpha} + \dfrac{a^\beta}{b^\alpha} + \dfrac{c^\beta}{a^\alpha} + \dfrac{a^\beta}{c^\alpha} +$ &c., $\dfrac{1}{a^\alpha b^\beta} + \dfrac{1}{a^\beta b^\alpha} + \dfrac{1}{a^\alpha c^\beta} + \dfrac{1}{a^\beta c^\alpha} +$ &c. from these coefficients, without knowing the roots.

SECTION XIII.

Prob. Reduce the numerical expression $(\alpha\beta\gamma)$, which contains three radical exponents, to a numerical expression containing no more than two radical exponents.

Solution. For the sake of perspicuity, I shall only begin with three roots a, b, c. If we multiply the equation

$$(\alpha\beta) = a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + b^\alpha c^\beta + b^\beta c^\alpha$$

by

$$(\gamma) = a^\gamma + b^\gamma + c^\gamma,$$

we then obtain

$(\gamma)(\alpha\beta) =$
$a^{\alpha+\gamma}b^{\beta} + a^{\beta}b^{\alpha+\gamma} + a^{\alpha+\gamma}c^{\beta} + a^{\beta}c^{\alpha+\gamma} + b^{\alpha+\gamma}c^{\beta} + b^{\beta}c^{\alpha+\gamma} +$
$a^{\beta+\gamma}b^{\alpha} + a^{\alpha}b^{\beta+\gamma} + a^{\beta+\gamma}c^{\alpha} + a^{\alpha}c^{\beta+\gamma} + b^{\beta+\gamma}c^{\alpha} + b^{\alpha}c^{\beta+\gamma} +$
$a^{\alpha}b^{\beta}c^{\gamma} + a^{\alpha}b^{\gamma}c^{\beta} + a^{\beta}b^{\alpha}c^{\gamma} + a^{\beta}b^{\gamma}c^{\alpha} + a^{\gamma}b^{\alpha}c^{\beta} + a^{\gamma}b^{\beta}c^{\alpha}.$

2. Now since the first row in the second part of this equation $= (\overline{\alpha+\gamma}\beta)$, the second $= (\overline{\beta+\gamma}\alpha)$, and the third $= (\alpha\beta\gamma)$, we consequently have

$(\psi) \ldots\ldots (\gamma)(\alpha\beta) = (\overline{\alpha+\gamma}\beta) + (\overline{\beta+\gamma}\alpha) + (\alpha\beta\gamma)$
and hence further

$(\alpha\beta\gamma) = (\gamma)(\alpha\beta) - (\overline{\alpha+\gamma}\beta) + (\overline{\beta+\gamma}\alpha)$
The functions $(\alpha\beta\gamma)$ is reduced to three others $(\alpha\beta)$, $(\overline{\beta+\gamma}\beta)$, $(\overline{\beta+\gamma}\alpha)$, each of which contains only two radical exponents, as required.

3. But since in this case only three roots a, b, c, were assumed, there may be a doubt as to the general application of the result so found. The following proof will remove this doubt.

4. In the first place, it is evident, that there are no equal terms in the product $(\gamma)(\alpha\beta)$. Take any term, for instance $b^{\beta+\gamma}d^{\alpha}$ of this product. This term can arise in no other way than by the multiplication of b^{γ} in (γ) by $b^{\beta}d^{\alpha}$ in $(\alpha\beta)$. If this term occurred more than once in the evolution of $(\gamma)(\alpha\beta)$, then $b^{\beta}d^{\alpha}$ must also occur more than once in $(\alpha\beta)$, which is impossible. But in the aggregate $(\overline{\alpha+\gamma}\beta) + (\overline{\beta+\gamma}\alpha) + (\alpha\beta\gamma)$, which constitutes the second term of the equation (ψ), there are not even two equal terms; this immediately appears from

the construction of the numerical expression, of which this aggregate consists.

5. But when the number of roots $= n$, the numerical expression $(\alpha\beta)$ consists of $n \,.\, n-1$ terms (§ IV), and consequently the product $(\gamma)\ (\alpha\beta)$, of $n^2 \,.\, n-1$ terms. The number of terms, which the sums $(\overline{\alpha+\gamma\beta})$, $(\overline{\beta+\gamma\alpha})$, $(\alpha\beta\gamma)$ contain, are for each of the two first $= n \,.\, n-1$, and for the last $= n \,.\, n-1 \,.\, n-2$; consequently for the aggregate $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma) = 2n \,.\, n-1 + n \,.\, n-1 \,.\, n-2 = n^2 \,.\, n-1$. This aggregate $\therefore$ contains exactly as many terms as the product $(\gamma)\ (\alpha\beta)$.

6. Further I affirm, that in the aggregate $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma)$ there can be no term, which is not also in $(\gamma)\ (\alpha\beta)$: for if, for instance, the terms $c^{\alpha+\gamma}+d^{\beta}$, $b^{\gamma}c^{\beta}e^{\alpha}$ are not in $(\gamma)\ (\alpha\beta)$, then likewise the terms $c^{\alpha}d^{\beta}$, $c^{\beta}e^{\alpha}$ are not in $(\alpha\beta)$; which is impossible.

7. From these conclusions it follows: first, that the terms of each of the two functions $(\gamma)\ (\alpha\beta)$, $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma)$ are different from one another; secondly, that the number of terms in the one is the same as in the other; thirdly, that there can be no term in the second, which is not also contained in the first. Hence it evidently follows, that they must be the same, and that consequently the equation (ψ) is true for every number of roots.

SECTION XIV.

PROB. Reduce the numerical expression $(\alpha\beta\gamma\delta)$, with four radical exponents, to another, in which there at most only three radical exponents.

Solution 1. If we examine the equations (χ) and (ψ) § XII and XIII, we are justified by analogy in making the following hypothesis:

$$(\delta)\ (\alpha\beta\gamma) =$$

$$(\overline{\alpha + \delta}\beta\gamma) + (\overline{\beta + \delta}\alpha\gamma) + (\overline{\gamma + \delta}\alpha\beta) + (\alpha\beta\gamma\delta).$$

In order to prove that this hypothesis is allowable, we can proceed in the same way as in the foregoing Section.

2. In the first place it may be proved, that in the function $(\delta)\ (\alpha\beta\gamma)$, no term can occur more than once. For if, for instance, the term $b^{\alpha+\delta}+c^{\gamma} f^{\beta}$, or $a^{\alpha} c^{\beta} d^{\gamma} e^{\delta}$ is oftener contained in this function, then also must $b^{\alpha} c^{\gamma} f^{\beta}$, or $a^{\alpha} c^{\beta} d^{\gamma}$, be oftener found in $(\alpha\beta\gamma)$, which is impossible. But from the nature of the function $(\overline{\alpha+\delta}\beta\gamma)+(\overline{\beta + \delta}\alpha\gamma) + (\overline{\gamma + \delta}\alpha\beta) + (\alpha\beta\gamma\delta)$ it can contain no term more than once. Consequently the terms of both functions are all different from one another.

3. The number of terms in $(\alpha\beta\gamma) = n \,.\, n-1 \,.\, n-2$ (§ IV), consequently those in the function $(\delta)\ (\alpha\beta\gamma) = n^2 \,.\, n - 1 \,.\, n - 2$. The number of terms in the function $(\overline{\alpha + \delta}\beta\gamma) + (\overline{\beta + \delta}\alpha\gamma) + (\overline{\gamma + \delta}\alpha\beta) + (\alpha\beta\gamma\delta = 3 \times n \,.\, n-1 \,.\, n-2 + n \,.\, n-1 \,.\, n-2 \,.\, n-3 = n^2 \,.\, n-1 \,.\, n-2$. Both functions $\therefore$ have the same number of terms.

4. Further, in the function $[\overline{\alpha + \delta}\beta\gamma] + [\overline{\beta + \delta}\alpha\gamma] + [\overline{\gamma + \delta}\alpha\beta] + [\alpha\beta\gamma\delta]$, there can be no term which is not contained also in the function $[\delta]\ [\alpha\beta\gamma]$. For if, for instance, the term $b^{\alpha} d^{\gamma} e^{\beta+\delta}$, or $a^{\gamma} b^{\beta} e^{\alpha} f^{\delta}$, in the first function, be not also contained in the second, ∴ also $b^{\alpha} \delta^{\gamma} e^{\beta}$, or $a^{\gamma} b^{\beta} e^{\alpha}$ is not in $[\alpha\beta\gamma]$; which is impossible.

5. From 2, 3, 4, it may be inferred, as in the foregoing section, that the two assumed functions must be equal, and that consequently the assumed equation is correct.

6. But from this equation we obtain
$[\alpha\beta\gamma\delta] =$
$[\delta]\ [\alpha\beta\gamma] - [\overline{\alpha + \delta}\beta\gamma] - [\overline{\beta + \delta}\alpha\gamma] - [\overline{\gamma + \delta}\alpha\beta]$
which verifies the problem, because in the second part of this equation, besides $[\delta]$, there are only numerical expressions with three radical exponents.

SECTION XV.

Prob. Reduce the general numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \iota\kappa\lambda]$ with m radical exponents, to others, which contain at most $m - 1$ radical exponents.

Solution 1. From §§ XII, XIII and XIV, we have sufficient reasons for assuming the following equations:
$[\lambda]\ [\alpha\beta\gamma\delta \ldots \iota\kappa] =$
$[\overline{\alpha + \lambda}\beta\gamma\delta \ldots \iota\kappa] + [\overline{\beta + \lambda}\alpha\gamma\delta\, \iota\kappa] + [\overline{\gamma + \lambda}\alpha\beta\delta \ldots \iota\kappa] + \ldots\ldots + [\overline{\kappa + \lambda}\alpha\beta\gamma\delta \ldots \iota] + [\alpha\beta\gamma\delta \ldots \iota\kappa\lambda]$;
the accuracy of which may very easily be proved.

2. For, in the first place, it may be proved by a similar manner as that used in §§ XIII and XIV, that both the function in the first part, and that in the second part of this equation, contain terms widely different from one another, and that for each term in the second of these two functions, there must be one equal to it in the first.

3. Further, since all numerical expressions in the equation contain $m - 1$ radical exponents, $[\lambda]$ and $[\alpha\beta\gamma\delta \ldots \iota\kappa\lambda]$ excepted, of which the last contains m radical exponents; $\therefore$ by § IV, the number of the terms in the function of the first part of this equation

$$= n \times n \,.\, n - 1 \,.\, n - 2 \,\ldots\ldots\, n - m + 2$$

and in the function of the second part

$$= m - 1 \times n \,.\, n - 1 \,.\, n - 2 \,\ldots\ldots\, n - m + 2 +$$
$$n \,.\, n - 1 \,.\, n - 2 \,\ldots\ldots\, n - m + 2 \,.\, n - m + 1$$
$$= n \times n \,.\, n - 1 \,.\, n - 2 \,\ldots\ldots\, n - m + 2.$$

These two functions consequently consist of the same number of terms.

4. From 2 and 3 we may infer, in the same manner as in §§ XIII and XIV the accuracy of the assumed equation. From this equation, however, we obtain

$$(\Theta) \ldots\ldots [\alpha\beta\gamma\delta \ldots \iota\kappa\lambda] = [\lambda]\, [\alpha\beta\gamma\delta \ldots \iota\kappa]$$
$$- [\overline{\alpha + \lambda}\beta\gamma\delta \ldots \iota\kappa] - [\overline{\beta + \lambda}\alpha\gamma\delta \ldots \iota\kappa]$$
$$- \ldots\ldots\ldots - [\overline{\kappa + \lambda}\alpha\beta\gamma\delta \ldots \iota]$$

which answers the condition of the problem.

Remark. The formula (Θ) obtains both for positive

and negative radical exponents, because the conclusions remain the same when the signs are changed. By means of this formula we are enabled to reduce any numerical expression $[\alpha\beta\gamma\delta \ldots \lambda]$ to others, which contain one radical exponent less; and if we continue the operation with this diminished radical exponent, we at length arrive at sums of powers only, which, whether the exponents be positive or negative, may again be always expressed, by §§ VIII and XI, by the coefficients of the equation, to which the numerical expression refers.

SECTION XVI.

Hitherto it has been assumed, that the radical exponents in the numerical expression $[\alpha\beta\gamma\delta \ldots \lambda]$ are all different from one another. If this be not the case, and the expression is consequently of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots \kappa^{\mathfrak{k}}]$, then the preceding formulæ, if we wish to apply them further, must undergo some modifications. It has already been shewn in § V, that if two numerical expressions $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots \kappa^{\mathfrak{k}}]$, $[\alpha\beta\gamma\delta \ldots \zeta]$, the first with, and the other without repeating exponents, contain the same number of radical exponents, the number of terms in the first is less than those in the second by $1 \,.\, 2 \ldots \mathfrak{a} \times 1 \,.\, 2 \ldots \mathfrak{b} \times 1 \,.\, 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 \,.\, 2 \ldots \mathfrak{k}$. The reason of this is only to be accounted for in this way, that in the case of equal radical exponents, there are exactly the same number of terms, which are equal to one another in the numerical expression $[\alpha\beta\gamma\delta \ldots \zeta]$ for each combination of the roots a, b, c, d, &c., of which terms, only one is retained in $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}} \ldots \kappa^{\mathfrak{k}}]$, as we already know from the rule of com-

binations. Hence, however, it follows, in order to adapt the formulæ already found to this case, that each expression of the form $[\alpha^{a}\beta^{b}\gamma^{c}\delta^{d}\ldots\kappa^{k}]$ must be multiplied by the number $1.2\ldots a\times 1.2\ldots b\times 1.2\ldots c\times\ldots\ldots\times 1.2\ldots k$, &c. whose magnitude depends upon the repeating exponents a, b, c, d, &c. For shortness' sake, in the course of the calculation, I shall denote this factor or coefficient by κ, and when there are more, by κ, κ', κ'', &c.

SECTION XVII.

PROB. To reduce the numerical expression $[\alpha^{a}]$, with a equal radical exponents, to others, which contain only $a-1$ radical exponents.

Solution 1. Assume that the numerical expression $[\alpha\beta\gamma\delta\ldots\iota\kappa\lambda]$ in the equation (Θ), § XV contains a radical exponents, and that $\alpha=\beta=\gamma=\delta=\ldots=\lambda$; then this expression is changed to $[\alpha^{a}]$; further, the product $[\lambda]\,[\alpha\beta\gamma\delta\ldots\iota\kappa]$ to $[\alpha]\,[\alpha^{a-1}]$, and all the remaining $a-1$ numerical expressions in the second part of this equation to $[\overline{2\alpha}\,\alpha^{a-2}]$. For the reasons given in the foregoing §, if the coefficients κ, κ', κ'', are prefixed to the numerical expressions, we then obtain,

$$\kappa.[\alpha^{a}]=\kappa'.[\alpha]\,[\alpha^{a-1}]-[a-1]\,\kappa''\,[\overline{2\alpha}\,\alpha^{a-2}]$$

2. But $\kappa=1.2.3\ldots a$, $\kappa'=1.2.3\ldots a-1$, $\kappa''=1.2.3\ldots a-2$. By substituting these values, and then dividing by $1.2.3\ldots a-1$, we then get

$$a\,[\alpha^{a}]=[\alpha]\,[\alpha^{a-1}]-[\overline{2\alpha}\,\alpha^{a-2}].$$

Consequently by this equation $[\alpha^{a}]$ is reduced to the

numerical expressions $[\alpha^{\mathfrak{a}-1}]$, $[\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}]$, each of which contains only $\mathfrak{a}-1$ radical exponents.

SECTION XVIII.

Prob. Reduce the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}]$, with $\mathfrak{a}+\mathfrak{b}$ radical exponents, to others, which contain one radical exponent less.

Solution 1. From the equation (Θ), § XV, when we put $\mathfrak{a}$ of the radical exponents $= \alpha$, and the $\mathfrak{b}$ remaining ones $= \beta$, we obtain

$$\kappa\,.\,[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}] =$$
$$\kappa'\,.\,[\alpha]\,[\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}] - [\mathfrak{a}-1]\,.\,\kappa''\,.\,[\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}]$$
$$- \mathfrak{b}\,.\,\kappa'''\,.\,[\overline{\alpha+\beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}]$$

2. But (§ XVI)

$$\begin{aligned}
\kappa &= 1\,.\,2\,.\,3\ldots\mathfrak{a} \quad\times 1\,.\,2\,.\,3\ldots\mathfrak{b}\\
\kappa' &= 1\,.\,2\,.\,3\ldots\mathfrak{a}-1 \times 1\,.\,2\,.\,3\ldots\mathfrak{b}\\
\kappa'' &= 1\,.\,2\,.\,3\ldots\mathfrak{a}-2 \times 1\,.\,2\,.\,3\ldots\mathfrak{b}\\
\kappa''' &= 1\,.\,2\,.\,3\ldots\mathfrak{a}-1 \times 1\,.\,2\,.\,3\ldots\mathfrak{b}-1
\end{aligned}$$

If we substitute these values, and then divide by $1\,.\,2\,.\,3\ldots\mathfrak{a}-1 \times 1\,.\,2\,.\,3\ldots\mathfrak{b}$, we obtain

$$\mathfrak{a}\,.\,[\kappa^{\mathfrak{a}}\beta^{\mathfrak{b}}] =$$
$$[\alpha]\,[\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}] - [\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}] - [\overline{\alpha+\beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}]$$

which was required.

SECTION XIX.

Prob. Reduce the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}]$, with $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}$ radical exponents, to others, which contain one radical exponent less.

Solution 1. From the equation (Θ), § XV., when we put $\mathfrak{a}$ of the radical exponents $= \alpha$, $\mathfrak{b}$ of them $= \beta$, and the remaining ones $\mathfrak{c} = \gamma$, we get

$$\kappa \,.\, [\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] =$$
$$\kappa' \,.\, [\alpha]\, [\alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] - [\mathfrak{a}-1] \,.\, \kappa'' \,.\, [\overline{2\alpha}\, \alpha^{\mathfrak{a}-2} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}]$$
$$- \mathfrak{b} \,.\, \kappa''' \,.\, [\overline{\alpha + \beta}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}-1} \gamma^{\mathfrak{c}}]$$
$$- \mathfrak{c} \,.\, \kappa^{IV} [\overline{\alpha + \gamma}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}-1}]$$

2. But

$$\kappa \;\; = 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{a} \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa' \;\; = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa'' \;\; = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-2 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa''' = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{b}-1 \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa^{IV} = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}-1.$$

If we substitute these values, and then divide by $1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$ we obtain

$$\alpha\, [\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] = [\alpha]\, [\alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] - [\overline{2\alpha}\, \alpha^{\mathfrak{a}-2} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}]$$
$$- [\overline{\alpha + \beta}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}-1} \gamma^{\mathfrak{c}}] - [\overline{\alpha + \gamma}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}-1}]$$

as required.

SECTION XX.

Prob. Reduce the general numerical expression $[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}} \delta^{\mathfrak{d}} \ldots \kappa^{\mathfrak{k}} \lambda^{\mathfrak{l}}]$, with $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} + \ldots + \mathfrak{k} + \mathfrak{l}$ radical exponents, to others, which contain one radical exponent less.

Solution 1. If we compare the operation in §§ XVIII, XIX, XX., we shall, with very little trouble, obtain from it the following general equation :

$$(\mathfrak{C}) \ldots\ldots\ldots \alpha\,[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] =$$

$$[\alpha]\,[\alpha^{\mathfrak{a}-1}\alpha^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$

$$-[\overline{\alpha+\beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{\alpha+\gamma}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots - [\overline{\alpha+\kappa}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}-1}\lambda^{\mathfrak{l}}]$$

$$-[\overline{\alpha+\lambda}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}-1}]$$

in which each of the numerical expressions in the second part contains no more than $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} + \ldots\ldots + \mathfrak{k} + \mathfrak{l} - 1$ radical exponents.

The formula thus found obtains, as also the one in § XV, whether the radical exponents, α, β, γ, &c. be positive or negative, because in this case the conclusions remain the same.

For the particular case, in which $\mathfrak{a} = 1$, this formula ceases to be applicable, because the repeating exponents $\mathfrak{a}-1$, $\mathfrak{a}-2$, which are contained in it, then become 0 and -1, which is impossible. In this case we must make use of the following formula:

$$[\alpha\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] =$$

$$[\alpha]\,[\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{\alpha+\beta}\,\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$

$$-[\overline{\alpha+\gamma}\,\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots - [\overline{\alpha+\lambda}\,\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}-1}]$$

which may be derived from the same sources as the preceding.

Remark. By means of the equation ($\mathfrak{C}$) we are now enabled to reduce every numerical expression of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$, by a certain reduction of the repeating exponents, to another numerical expression of the form $[\alpha\beta\gamma\delta\ldots\kappa\lambda]$; and since this last, by means of the equation (Θ) in § XV, may always be reduced merely to the sums of powers, and ∴ may at length be expressed

by the coefficients of the given equation; consequently we are always enabled to express any numerical expression of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$ by the coefficients of the given equation.

Moreover, the equation (☾) is always true, so long as we assign no determinate values to the radical exponents α, β, γ, δ, &c. For determinate values, it may happen that radical exponents are equal to one another, which in the general expression were considered as different: thus, for instance, when in the equation (☾) $2\alpha = \beta$, or $\alpha + \beta = \gamma$. In such cases as these, we shall do well, in order to avoid mistakes, to add the following equation derived from (Θ), § XV:

$$\begin{aligned}
\kappa \,.\, [\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] = \kappa' \,.\, [\alpha]\,[\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] \\
- [\mathfrak{a}-1] \,.\, \kappa'' \,.\, [\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] \\
- \mathfrak{b} \,.\, \kappa''' \,.\, [\overline{\alpha + \beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] \\
- \mathfrak{c} \,.\, \kappa^{IV} \,.\, [\overline{\alpha + \gamma}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] \\
\text{\&c.}
\end{aligned}$$

in which the coefficients, κ, κ', κ'', κ''', &c., have the values given in § XVI.

Every integral or fractional rational symmetrical function of the roots a, b, c, d, &c., however constituted, must necessarily be composed of numerical expressions of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\lambda^{\mathfrak{l}}]$. Now since these last, as we have already seen, can always be expressed rationally by the coefficients of the equation to which they relate, consequently also the former can always be expressed rationally by these coefficients; to prove which was the aim of the present chapter.

Since, however, the rule of symmetrical functions is of the greatest importance in the theory of equa-

tions, it is often requisite to express these functions by the coefficients of the given equation, I have ∴ subjoined three tables, in which all numerical expressions, in which the sum of the radical exponents does not exceed the number 10, are fully calculated. Thus Table I* contains, in six small tables, the values of all numerical expressions for the Nos. 2, 3, 4, 5, 6, 7; Table II, those for the Nos. 8 and 9; and Table III, those for the No. 10. The arrangement of these tables is evident at first sight. The letters A, B, C, D, &c. are the coefficients of the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \&c. = 0$, which is the basis of the numerical expressions, and these last, for the sake of facilitating the calculation, are assumed with alternate signs. In the horizontal lines, the numerical expressions themselves are found in a combination series; in the first upper horizontal series, are the different terms of their values, and in the vertical columns under them the numerical coefficients belonging to each term, according to the difference of the numerical expressions. Where there are terms wanting, or the numerical coefficients $= 0$, asterisks are placed. Thus, for instance, in Table III $[1^2 4^2] = B^3D - 3\,ABCD + 3\,C^2D + 3\,A^2D^2 - 3\,BD^2 - AB^2E + 2\,A^2CE + BCE - 8\,ADE + 5\,E^2 + A^2BF - B^2F - 3\,ACF + 9\,DF - 6\,A^3G + 17\,ABG - 15\,CG + 6\,A^2H - 11\,BH - 6\,AI + 15\,K.$

These tables were calculated by means of the equations (Θ) and (ↄ) in § XV. and § XX. For the more easy application of these tables, it is, however, necessary that the calculation be made successively, and

* *Note.*—These tables are to be found at the end. *Translator.*

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that, in order to find the numerical expressions for any determinate sum of theadical exponents, we should first know all those for lower sums. Likewise the sums of powers must be previously calculated by means of the equations in 8, § VIII, which equations, on account of the change of the signs in the assumed equation, have the following values:

$$(1) = A$$
$$(2) = A^2 - 2B$$
$$(3) = A^3 - 3AB + 3C$$
$$(4) = A^4 - 4A^2B + 4AC + 2B^2 - 4D$$
&c.

Thus, for the successive calculation of the numerical expressions, Table III, we have the following equations:

$$
\begin{aligned}
(\overline{10}) &= (\overline{10}) \\
(19) &= (1)(9) - (\overline{10}) \\
(28) &= (2)(8) - (\overline{10}) \\
(37) &= (3)(7) - (\overline{10}) \\
(46) &= (4)(6) - (\overline{10}) \\
2\,(5^2) &= (5)(5) - (\overline{10}) \\
2\,(1^2 8) &= (1)(18) - (28) - (19) \\
(127) &= (1)(27) - (37) - (28) \\
(136) &= (1)(36) - (46) - (37) \\
(145) &= (1)(45) - 2\,(5^2) - (46) \\
2\,(2^2 6) &= (2)(26) - (46) - (28)
\end{aligned}
$$

The numerical expressions in the first part of the equations, depend here, as is easily seen, either on the foregoing, or on such numerical expressions as have a less sum of radical exponents, which, when these last are already found, may successively be determined.

II.—Complete solution of the symmetrical functions of the roots of an equation.

Section XXII.

TO solve a symmetrical function, here means no more than to find an expression for it, which contains only sums of powers.

A compound radical exponent implies one, which is compounded of more than one compound one, as $\alpha + \beta + \gamma + \delta +$ &c. in $(\alpha+\beta+\gamma+\delta+$&c.), or $(a\alpha+b\beta+c\gamma+d\delta$ +&c.) in $(a\alpha+b\beta+c\gamma+d\delta+$&c.) The terms of the first are α, β, γ, δ, &c.; the terms of the latter are $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, &c. In the opposite case, α and $a\alpha$, in (α), $(a\alpha)$, are simple radical exponents.

In order to show, that a numerical expression, viz. (α), can be raised to any power μ, I shall merely write $(\alpha)^\mu$. We must then very carefully transform $(\alpha)^\mu$ into (α^μ); for $(\alpha)^\mu = (\alpha)\ (\alpha)\ (\alpha)\ (\alpha) \ldots$; on the other hand, $(\alpha^\mu) = (\alpha\ \alpha\ \alpha \ldots)$. So in like manner $(3\alpha+2\beta)^\mu$ denotes the μth power of $(3\alpha+2\beta)$, and $(a\alpha+b\beta+c\gamma+d\delta+$&c.$)^\mu$ the μth power of $(a\alpha + b\beta + c\gamma + d\delta +$ &c.)

Section XXIII.

Prob. Represent the numerical expressions $(\alpha\beta)$, $(\alpha\beta\gamma)$, $(\alpha\beta\gamma\delta)$, &c. fully developed.

Solution 1. From § XII we immediately have

$$(\alpha\beta) = (\beta)\,(\alpha) - (\beta + \alpha)$$

2. From § XIII we at first obtain

$$(\alpha\beta\gamma) = (\gamma)\,(\alpha\beta) - (\overline{\gamma + \alpha}\beta) - (\overline{\gamma + \beta}\alpha).$$

But from 1 we have, when first $\gamma+\alpha$ is put for α, and afterwards $\gamma+\beta$ for β,

$$(\overline{\gamma + \alpha}\beta) = (\beta)\,(\gamma + \alpha) - (\gamma + \beta + \alpha)$$
$$(\overline{\gamma + \beta}\alpha) = (\gamma + \beta)\,(\alpha) - (\gamma + \beta + \alpha)$$

If these values, together with the value of $(\alpha\beta)$ from 1, be substituted in the foregoing equation, we then obtain

$$(\alpha\beta\gamma) = (\gamma)\,(\beta)\,(\alpha) - (\gamma)\,(\beta+\alpha) - (\gamma+\beta)\,(\alpha) - (\gamma+\alpha)\,(\beta) + 1\,.\,2\,(\gamma+\beta+\alpha)$$

3. From § XIV, we have

$$(\alpha\beta\gamma\,\delta) = (\delta)\,(\alpha\beta\gamma) - (\overline{\delta + \alpha}\beta\gamma) - (\overline{\delta + \beta}\alpha\gamma) - (\overline{\delta + \gamma}\alpha\beta)$$

In order to find the numerical expressions $(\overline{\delta + \alpha}\beta\gamma)$, $(\overline{\delta + \beta}\alpha\gamma)$, $(\overline{\delta + \gamma}\alpha\beta)$, we need only successively substitute $\delta+\alpha$ for α, afterwards $\delta+\beta$ for β; and lastly $\delta+\gamma$ for γ in the last equation in 2. If, after this, we substitute the values thus obtained, together with the value of $(\alpha\beta\gamma)$, we obtain

$$\begin{aligned}(\alpha\beta\gamma\delta) = \\ &(\delta)\,(\gamma)\,(\beta)\,(\alpha) - (\delta)\,(\gamma)\,(\beta+\alpha) - (\delta)\,(\gamma+\beta)\,(\alpha) \\ &-(\delta)\,(\gamma+\alpha)\,(\beta) + 1\,.\,2\,(\delta)\,(\gamma+\beta+\alpha) - (\delta+\gamma)\,(\beta)\,(\alpha) \\ &-(\delta+\beta)\,(\gamma)\,(\alpha) - (\delta+\alpha)\,(\gamma)\,(\beta) + (\delta+\gamma)\,(\beta+\alpha) \\ &+ 1.2\,(\delta+\beta+\alpha)\,(\gamma) + 1\,.\,2\,(\delta+\gamma+\beta)\,(\alpha) + (\delta+\alpha)(\gamma+\beta) \\ &+ 1.2\,(\delta+\gamma+\alpha)\,(\beta) + (\delta+\beta)(\gamma+\alpha) - 1.2.3(\delta+\gamma+\beta+\alpha).\end{aligned}$$

4. So, in like manner, from § XV, we have

$$(\alpha\beta\gamma\delta\varepsilon) = (\varepsilon)\,(\alpha\beta\gamma\delta) - (\overline{\varepsilon+\alpha}\beta\gamma\delta) - (\overline{\varepsilon+\beta}\alpha\gamma\delta)$$
$$- (\overline{\varepsilon+\gamma}\alpha\beta\delta) - (\overline{\varepsilon+\delta}\alpha\beta\gamma).$$

We get the values of $(\overline{\varepsilon+\alpha}\beta\gamma\delta)$, $(\overline{\varepsilon+\beta}\alpha\gamma\delta)$, $(\overline{\varepsilon+\gamma}\alpha\beta\delta)$, $(\overline{\varepsilon+\delta}\alpha\beta\gamma)$ completely developed from the last equation in 3, by substituting in it successively $\varepsilon+\alpha$ for α, $\varepsilon+\beta$ for β, $\varepsilon+\gamma$ for γ, and $\varepsilon+\delta$ for δ. The substitution of these values, together with that of $(\alpha\beta\gamma\delta)$ in the foregoing equation, gives the solution required.

5. In this way we could proceed further, since we always go from one solution to another, and thus find the solutions of the numerical expressions, which contain six, seven, eight, &c. radical exponents.

6. Generally, if we have already found the solution of a numerical expression $(\alpha\beta\gamma\delta\ldots\kappa)$, and wish from hence to derive the solution of another $(\alpha\beta\gamma\delta\ldots\kappa\lambda)$, which contains λ more radical exponents, we must, in the first place, multiply merely the solution of $(\alpha\beta\gamma\delta\ldots\kappa)$ by (λ), then in this solution substitute throughout, first $\lambda+\alpha$ for α, then $\lambda+\beta$ for β, $\lambda+\gamma$ for γ, &c. and change the signs of the results and the former product.

SECTION XXIV.

Prob. Find the law, by which the terms in the solutions of $(\alpha\beta)$, $(\alpha\beta\gamma)$, $(\alpha\beta\gamma\delta)$, &c. are formed, when the coefficients and the signs are left out.

Solution 1. If in the solutions of the above numerical

expressions in the foregoing §, we omit the brackets and the signs, and separate the radical exponents, which belong to the different numerical expressions in each term, by a comma, and all the terms by a semicolon, we then, by means of (a), find the following:

I. α

II. $\beta, \alpha; \beta+\alpha$

III. $\gamma, \beta, \alpha; \gamma, \beta+\alpha; \gamma+\beta, \alpha; \gamma+\alpha, \beta; \gamma+\beta+\alpha$

IV. $\delta, \gamma, \beta, \alpha; \delta, \gamma, \beta+\alpha; \delta, \gamma+\beta, \alpha; \delta, \gamma+\alpha, \beta$
$\delta, \gamma+\beta+\alpha; \delta+\gamma, \beta, \alpha; \delta+\beta, \gamma, \alpha; \delta+\alpha, \gamma, \beta$
$\delta+\gamma, \beta+\alpha; \delta+\beta+\alpha, \gamma; \delta+\gamma+\beta, \alpha; \delta+\alpha, \gamma+\beta$
$\delta+\gamma+\alpha, \beta; \delta+\beta, \gamma+\alpha; \delta+\gamma+\beta+\alpha$

&c.

2. Hence we may perceive at first sight and from 6 in the foregoing § the law of the successive formation of the terms. Thus, in order to derive the terms of a solution from the terms of the immediately preceding one, we must

(a) put before each of all the terms of the preceding solution, the radical exponent which is now to be added;

(b) we must connect this by the sign + with each radical exponent of every term, while, at the same time, we add the remaining radical exponents of the same term without any change.

Thus, for instance, if we wish to derive IV from III, by the rule (a) we get

$\delta, \gamma, \beta, \alpha; \delta, \gamma, \beta+\alpha; \delta, \gamma+\beta, \alpha; \delta, \gamma+\alpha, \beta; \delta, \gamma+\beta+\alpha$

and by the rule (b) from the first term in III

$\delta + \gamma, \beta, \alpha$; $\delta + \beta, \gamma, \alpha$; $\delta + \alpha, \gamma, \beta$;

from the second term in III

$\delta + \gamma, \beta + \alpha$; $\delta + \beta + \alpha, \gamma$;

from the third term in III

$\delta + \gamma + \beta, \alpha$; $\delta + \alpha, \gamma + \beta$;

from the fourth term in III

$\delta + \gamma + \alpha, \beta$; $\delta + \beta, \gamma + \alpha$;

lastly from the fifth term in III

$\delta + \gamma + \beta + \alpha$.

The foundation of this method is so evident from the foregoing §, that it requires no further explanation.

3. But since this mode of representation possesses this disadvantage, that in finding the following solutions, we must first add the foregoing, we can ∴ with great advantage make use of the Hindenburgian method of involution, which is already known to my readers from the first principles of the rule of combination. Here follows this involution, whose construction is immediately deducible from 2 :

$$\begin{array}{l|l|ll}
\delta & \gamma & \beta & \alpha \\
\delta & \gamma & \beta & + \alpha \\
\hline
\delta & \gamma + \beta, & & \alpha \\
\delta & \gamma + \alpha, & & \beta \\
\delta & \gamma + \beta & & + \alpha \\
\hline
\end{array}$$

$$\begin{array}{llll}
\&c.\ \delta + \gamma, & \beta, & & \alpha \\
\delta + \beta, & \gamma, & & \alpha \\
\delta + \alpha, & \gamma, & & \beta \\
\delta + \gamma, & \beta, & + & \alpha \\
\delta + \beta + \alpha, & & & \gamma \\
\delta + \gamma + \beta, & & & \alpha
\end{array}$$

$$\begin{array}{l} \delta+\alpha,\ \ \gamma\ +\ \beta \\ \delta+\gamma+\alpha,\ \ \ \ \beta \\ \delta+\beta,\ \ \gamma\ +\ \alpha \\ \delta+\gamma+\beta\ \ +\ \alpha \\ \quad\&c. \end{array}$$

It is not necessary, in the first place, to remind my readers, that this mode of representation, besides the advantage that it immediately gives what is sought, possesses also this one, that each solution includes all the foregoing, as the brackets denote, and this follows of course from the very nature of involution.

REMARK. The involution which is here given, includes, besides, as may easily be observed, all possible combinations of the radical exponents α, β, γ, δ, &c. both simple and compound, consequently can be used with advantage in many other cases, in which it is required to find all the possible combinations of this kind in a given number of things.

SECTION XXV.

PROB. Find the law of the coefficients and the signs in the solution of the numerical expression $(\alpha\beta\gamma\delta\ldots\lambda)$.

Solution 1. From the method in § XXIII, by which the solutions are derived from one another, and from the results themselves, it may, with some reason, be presumed, that the coefficients of the terms and their signs are subject to the following laws:

(a) That each numerical expression of a simple radical exponent has unity for its coefficient;

(*b*) That each numerical expression of a compound radical exponent of m terms has the coefficient $1 \cdot 2 \cdot 3 \ldots\ldots m-1$;

(*c*) That the sign $-$ or $+$ may be given to every numerical expression whatever, according as the number of the terms of its radical exponent is odd or even.

Thus if these rules be correct, the term (α) $(\beta+\gamma)$ $(\delta+\varepsilon+\zeta)$ $(\eta+\vartheta+\iota+\kappa)$ has the coefficient $1 \times 1 \times 1 \cdot 2 \times 1 \cdot 2 \cdot 3$, or merely $1 \cdot 2 \times 1 \cdot 2 \cdot 3$, with the sign $+$, because it has two radical exponents of an even, and two of an odd, number of terms.

That these rules are correct for $(\alpha\beta)$, $(\alpha\beta\gamma)$, $(\alpha\beta\gamma\delta)$, one can be readily convinced of by inspection. It only remains now, by a very common method in mathematics, to prove the rule, that when they obtain for any solution, they likewise necessarily obtain for the following one.

3. For instance, let

$$(A) \ldots\ldots (\alpha+\beta+\gamma+\ldots+\kappa)\ (\lambda+\mu+\nu\ldots+\psi)$$

be any term in the solution of $(\alpha\beta\gamma \ldots\ldots \psi)$. The radical exponent of the first numerical expression in (A) contains m terms, that of the second n terms. Consequently the coefficient of the product, according to the hypothesis $= 1 \cdot 2 \cdot 3 \ldots\ldots m-1 \times 1 \cdot 2 \cdot 3 \ldots\ldots n-1$.

4. Now let $(\alpha\beta\gamma\ldots\psi\,\omega)$ be another numerical expression, which contains more radical exponents than the preceding by ω. For its solution, the term (A) by 6, § XXIII gives the three following terms:

$$(\omega)\,(\alpha+\beta+\gamma+\ldots+\kappa)\,(\lambda+\mu+\nu+\ldots+\psi)$$
$$-\,(\alpha+\beta+\gamma+\ldots+\kappa+\omega)\,(\lambda+\mu+\nu+\ldots+\psi)$$
$$-\,(\alpha+\beta+\gamma+\ldots+\kappa)\,(\lambda+\mu+\nu+\ldots+\psi+\omega)$$

5. The first of these terms is obtained from the term (A), by multiplying the latter by (ω), and consequently it has the same coefficient and the same sign; which agrees with the hypothesis.

6. The second term in 4 arises from the substitution of $\omega+\alpha$, $\omega+\beta$, $\omega+\gamma\ldots\ldots\omega+\kappa$ for α, β, γ, $\ldots\ldots\kappa$, (6. § XXIII) and occurs m times. In like manner the third term arises from the substitution of $\omega+\lambda$, $\omega+\mu$, $\omega+\nu$, $\ldots\ \omega+\psi$ for λ, μ, ν, $\ldots\ldots\psi$, and $\therefore$ occurs n times. Consequently the second term must contain m, and the third n, more coefficients than the term (A).

7. Hence it follows, that the coefficient of the second term in 4. $= 1.2.3\ldots.m \times 1.2.3\ldots n-1$, and the coefficient of the third term $= 1.2.3\ldots.m-1 \times 1.2.3\ldots.n$. Likewise these terms have a different sign, from that of the term (A).

8. Since this agrees with the hypothesis, so it may be concluded, that, when the hypothesis is true for the term (A), it is necessarily true for the terms derived from it in the following solution.

9. Although here the term (A) has only been assumed as a product of two numerical expressions, it is sufficiently evident from the manner in which the proof has

been managed, that it may be extended to any other number of factors.

¶ 10. If ∴ the solution of $[\alpha\beta\gamma\ldots\psi]$ be subject to the assumed rules, so likewise is that of $[\alpha\beta\gamma\ldots\psi\omega]$ which follows it. But they obtain for the first solutions, they ∴ also obtain for all those which follow.

Corollary. In order ∴ to exhibit a numerical expression of the form $[\alpha\beta\gamma\ldots\lambda]$, in which all the radical exponents are different, completely solved, we only require to perform the involution given in § XXIV, and affix to each term the coefficient determined from 1 of this § with its sign. The following example will serve as an illustration.

Example. The complete solution of $[\alpha\beta\gamma\delta\varepsilon]$, when the brackets are omitted in the terms, is as follows:

$+1,\ \varepsilon \vert \delta \vert \gamma \vert \beta \vert \alpha$	$+2,\ \varepsilon+\gamma+\alpha,\ \delta,\ \beta$
$-1,\ \varepsilon \vert \delta \vert \gamma \vert \beta+\alpha$	$+1,\ \varepsilon+\beta,\ \delta,\ \gamma+\alpha$
$-1,\ \varepsilon \vert \delta \vert \gamma+\beta,\ \alpha$	$-2,\ \varepsilon+\delta,\ \gamma+\beta+\alpha$
$-1,\ \varepsilon \vert \delta \vert \gamma+\alpha,\ \beta$	$-6,\ \varepsilon+\gamma+\beta+\alpha,\ \delta$
$+2,\ \varepsilon \vert \delta \vert \gamma+\beta+\alpha$	$+2,\ \varepsilon+\delta+\gamma,\ \beta,\ \alpha$
$-1,\ \varepsilon \vert \delta+\gamma,\ \beta,\ \alpha$	$+1,\ \varepsilon+\beta,\ \delta+\gamma,\ \alpha$
$-1,\ \varepsilon \vert \delta+\beta,\ \gamma,\ \alpha$	$+1,\ \varepsilon+\alpha,\ \delta+\gamma,\ \beta$
$-1,\ \varepsilon \vert \delta+\alpha,\ \gamma,\ \beta$	$+2,\ \varepsilon+\delta+\beta,\ \gamma,\ \alpha$
$+1,\ \varepsilon \vert \delta+\gamma,\ \beta+\alpha$	$+1,\ \varepsilon+\gamma,\ \delta+\beta,\ \alpha$
$+2,\ \varepsilon \vert \delta+\beta+\alpha,\ \gamma$	$+1,\ \varepsilon+\alpha,\ \delta+\beta,\ \gamma$
$+2,\ \varepsilon \vert \delta+\gamma+\beta,\ \alpha$	$+2,\ \varepsilon+\delta+\alpha,\ \gamma,\ \beta$
$+1,\ \varepsilon \vert \delta+\alpha,\ \gamma+\beta$	$+1,\ \varepsilon+\gamma,\ \delta+\alpha,\ \beta$
$+2,\ \varepsilon \vert \delta+\gamma+\alpha,\ \beta$	$+1,\ \varepsilon+\beta,\ \delta+\alpha,\ \gamma$
$+1,\ \varepsilon \vert \delta+\beta,\ \gamma+\alpha$	$-2,\ \varepsilon+\delta+\gamma,\ \beta+\alpha$
$-6,\ \varepsilon \vert \delta+\gamma+\beta+\alpha$	$-2,\ \varepsilon+\beta+\alpha,\ \delta+\gamma$

$-1,\ \varepsilon+\delta,\ \gamma,\ \beta,\ \alpha$	$-6,\ \varepsilon+\delta+\beta+\alpha,\ \gamma$
$-1,\ \varepsilon+\gamma,\ \delta,\ \beta,\ \alpha$	$-2,\ \varepsilon+\gamma,\ \delta+\beta+\alpha$
$-1,\ \varepsilon+\beta,\ \delta,\ \gamma,\ \alpha$	$-6,\ \varepsilon+\delta+\gamma+\beta,\ \alpha$
$-1,\ \varepsilon+\alpha,\ \delta,\ \gamma,\ \beta$	$-2,\ \varepsilon+\alpha,\ \delta+\gamma+\beta$
$+1,\ \varepsilon+\delta,\ \gamma,\ \beta+\alpha$	$-2,\ \varepsilon+\delta+\alpha,\ \gamma+\beta$
$+1,\ \varepsilon+\gamma,\ \delta,\ \beta+\alpha$	$-2,\ \varepsilon+\gamma+\beta,\ \delta+\alpha$
$+2,\ \varepsilon+\beta+\alpha,\ \delta,\ \gamma$	$-6,\ \varepsilon+\delta+\gamma+\alpha,\ \beta$
$+1,\ \varepsilon+\delta,\ \gamma+\beta,\ \alpha$	$-2,\ \varepsilon+\beta,\ \delta+\gamma+\alpha$
$+2,\ \varepsilon+\gamma+\beta,\ \delta,\ \alpha$	$-2,\ \varepsilon+\delta+\beta,\ \gamma+\alpha$
$+1,\ \varepsilon+\alpha,\ \delta,\ \gamma+\beta$	$-2,\ \varepsilon+\gamma+\alpha,\ \delta+\beta$
$+1,\ \varepsilon+\delta,\ \gamma+\alpha,\ \beta$	$+24,\ \varepsilon+\delta+\gamma+\beta+\alpha$

We have $\therefore\ [\alpha\beta\gamma\delta\varepsilon] = [\varepsilon]\,[\delta]\,[\gamma]\,[\beta]\,[\alpha] -$
$[\varepsilon]\,[\delta]\,[\gamma]\,[\beta+\alpha] - [\varepsilon]\,[\delta]\,[\gamma+\beta]\,\alpha -$ &c.

SECTION XXVI.

If there are more radical exponents equal to one another in the numerical expression $[\alpha\beta\gamma \ldots\ldots \lambda]$, or if it takes the form $[\alpha^a \beta^b \gamma^c \ldots\ldots \kappa^k]$, the solution admits of reduction, because in this case more terms than one are equal to one another. By the application of the rules (a) and (b) in 2, § XXIV, we have only to take care that no term occurs more than once, and with this view it is only necessary in performing the involution, always to revert to the combinations already found, and to omit all those which occurred once before. Thus we find the involution for the terms of $[\alpha^3\beta^3]$ to be as follows:

$\beta \mid \beta \mid \beta \mid \alpha \mid \alpha \mid \alpha$	$2\beta,\ \beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta \mid \alpha \mid 2\alpha$	$2\beta,\ \beta+\alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta \mid 3\alpha$	$2\beta,\ \beta+2\alpha,\ \alpha$
$\beta \mid \beta \mid \beta+\alpha,\ \alpha,\ \alpha$	$2\beta,\ \beta+3\alpha$
$\beta \mid \beta \mid \beta+\alpha,\ 2\alpha$	$3\beta,\ \alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta+2\alpha,\ \alpha$	$3\beta,\ \alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta+3\alpha$	$3\beta,\ 3\alpha$
$\beta \mid 2\beta,\ \alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid 2\beta,\ \alpha,\ 2\alpha$	$\beta+\alpha,\ 2\beta+\alpha,\ \alpha$
$\beta \mid 2\beta,\ 3\alpha$	$\beta+\alpha,\ \beta+\alpha,\ \beta+\alpha$
$\beta \mid 2\beta+\alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ 2\alpha$
$\beta \mid \beta+\alpha,\ \beta+\alpha,\ \alpha$	$\beta+2\alpha,\ 2\beta+\alpha$
$\beta \mid 2\beta+\alpha,\ 2\alpha$	$2\beta+2\alpha,\ \beta+\alpha$
$\beta \mid \beta+2\alpha,\ \beta+\alpha$	$3\beta+2\alpha,\ \alpha$
$\beta \mid 2\beta+2\alpha,\ \alpha$	$3\beta+3\alpha$
$\beta \mid 2\beta+3\alpha$	

This involution contains every possible analysis of $3^{\alpha}+3^{\beta}$. In the same manner we obtain generally by the involution for $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$ every possible analysis of $\mathfrak{a}\alpha+\mathfrak{b}\beta+\mathfrak{c}\gamma+\ldots+\mathfrak{k}\kappa$. It may $\therefore$ be used with advantage in all those cases where it is required to represent analysis of this kind, with facility and without the danger of omission. In the numerical expression $[\alpha^{\mathfrak{a}}]$, the operation is merely reduced to a numerical analysis, respecting which information is given in the rule of combinations.

But as to the coefficients of the terms, it is easily conceived, that the rules (*a*) and (*b*), in 1, § XXV, in the case where the numerical expression $[\alpha\beta\gamma\ldots\lambda]$ is changed to $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\kappa^{\mathfrak{k}}]$, must undergo many modifications, and first, for this reason, because in this case

more terms than one of the solution vanish, secondly, for the reason given in § XVI. But in order that we may not be obliged in the sequel, to break the thread of the inquiry by extraneous matter, we shall premise with the following auxiliary rules.

SECTION XXVII.

I.—*AUXILIARY RULE.*

PROB. Find in how many ways a given number of things may be placed in a determinate number of divisions, in such a way, that in each of these divisions there is a given number of these things.

1. *Solution for two divisions.*

Let A be the number of all the things, a the number of these things which are to enter into the first of these divisions; $\therefore A-a$ the number in the second.

It is evident, that this is merely to find the number of combinations of the ath class in A things. Now this is

$$=\frac{A \cdot A-1 \cdot A-2 \ldots\ldots\ldots\ldots A-a+1}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a};$$

or if we multiply numerator and denominator by $1 \cdot 2 \cdot 3 \ldots A-a$, and then in the denominator substitute a' for $A-a$:

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots\ldots\ldots\ldots A-1 \cdot A}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots a \times 1 \cdot 2 \cdot 3 \ldots a'}$$

in which a and a' denote the numbers contained in the two divisions.

2. *Solution for three divisions.*

Again, let A be the number of things; further, a, a',

a'', the number in the first, second and third division, $\therefore a+a'+a''=A$.

But in the first division the number a may enter in as many different ways as there are combinations of the ath class in A things; the number of these ways is

$$=\frac{A\,.\,A-1\ldots\ldots\ldots\ldots A-a+1}{1\,.\,2\ldots\ldots\ldots\ldots\ldots\ldots a}$$

So in like manner the number of ways, in which the remaining $A-a$ things may be arranged in the second division in the number a'

$$=\frac{A-a\,.\,A-a-1\ldots\ldots\ldots A-a-a'+1}{1\,.\,2\ldots\ldots\ldots\ldots\ldots\ldots a'}.$$

Each of these last combinations may be associated with each of the first, and the number of combinations which is thus obtained is $\therefore$ the product of those two numbers. The number of ways, in which A things may be arranged in the first, second and third divisions, in the numbers a, a', a'' is consequently

$$=\frac{A\,.\,A-1\,.\,A-2\ldots\ldots\ldots\ldots A-a-a'+1}{1\,.\,2\,.\,3\ldots\ldots a\times 1\,.\,2\,.\,3\ldots\ldots\ldots\ldots a'},$$

or, when the numerator and denominator are multiplied by $1, 2\ldots a''=1\,.\,2\ldots\ldots A-a-a'$,

$$=\frac{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots\ldots A-1\,.\,A}{1\,.\,2\ldots\ldots a\times 1\,.\,2\ldots\ldots a'\times 1\,.\,2\ldots\ldots a''}.$$

3. *Solution for four divisions.*

Again, let A be the number of all the things, and a, a', a'', a''' the numbers which are separately contained in the first, second, third and fourth division, $\therefore a+a'+a''+a'''=A$. The number of cases, in which A things may be arranged in the number a,

$$=\frac{A\,.\,A-1\ldots\ldots\ldots\ldots A-a+1}{1\,.\quad 2\ldots\ldots\ldots\ldots\ldots\ldots a}$$

The number of cases, in which the remaining $A-a$ things may be arranged in the number a'

$$=\frac{A-a\,.\,A-a-1\ \ldots\ldots\ A-a-a'+1}{1\quad .\quad 2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ a'};$$

consequently the number of cases in which A things may be arranged, first in the number a, and then a'

$$=\frac{A\,.\,A-1\ \ldots\ldots\ldots\ldots\ldots\ A-a-a'+1}{1\,.\,2\ldots\ldots\ldots\ldots a\ \times\ 1\,.\,2\ldots\ldots\ldots\ldots a'},$$

The number of cases in which the number a'' may, in the third division, be arranged from the remaining $A-a-a'$ things

$$=\frac{A-a-a'\,.\,A-a-a'\,1\ldots\ldots A-a-a'-a''+1}{1\quad .\quad 2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ a''}.$$

Each of these cases may be combined with each of the preceding, ∴ the number of cases, in which the A things are arranged in the four divisions

$$=\frac{A\,.\,A-1\ \ldots\ldots\ldots\ldots\ldots\ A-a-a'-a''+1}{1\,.\,2\ldots a\ \times\ 1\,.\,2\ldots\ldots a'\ \times\ 1\,.\,2\ldots\ldots a''},$$

or, when we multiply numerator and denominator by $1\,.\,2\ \ldots\ldots\ a'''=1\,.\,2\ \ldots\ldots,\ A-a-a'-a''$

$$=\frac{1\,.\,2\,.\,3\,.\,4\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ A}{1\,.\,2\ldots a\times 1\,.\,2\ldots a'\times 1\,.\,2\ldots. a''\times 1\,.\,2\ldots. a'''}.$$

4. *General Solution.*

The conclusions drawn in 1, 2, 3, may easily be extended to any number of divisions. The number of ways in which A things may be arranged in n divisions, so that the first may contain the number a, the second

the number a', the third the number a'', &c.; lastly, the nth division the number $a^{(n-1)}$ of these things, is ∴

$$=\frac{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots A-1 \,.\, A}{1.2\ldots a \times 1.2\ldots a' \times 1.2\ldots a'' \times \ldots \times 1.2\ldots a^{(n-1)}},$$

Example. Place 16 balls in four divisions of 6, 5, 3, and 2: in how many ways can this be done?—Here $A=16$, $a=6$, $a'=5$, $a''=3$, $a'''=2$; consequently the required number

$$=\frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16}{1.2.3.4.5.6 \times 1.2.3.4.5 \times 1.2.3 \times 1.2}$$
$$= 20180160.$$

SECTION XXVIII.

II.—Auxiliary Rule.

Prob. The numbers A, B, C, &c. of different kinds of things are given: find in how many different ways these numbers may be arranged in a given number, when in each determinate division there shall be a given number of things of each kind.

Solution 1. For the sake of perspicuity, I shall assume the particular case, that there are only three numbers of a different kind given, which are to be arranged in four divisions, so that in the first division there are a things of the first, b things of the second, and c things of the third kind; and that a', b', c'; a'', b'', c''; a''', b''', c''', denote the same for the second, third, and fourth divisions, as a, b, c, do for the first, ∴ $a+a'+a''+a'''=A$, $b+b'+b''+b'''=B$, $c+c'+c''+c'''=C$. Further, let

$$\lambda = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A-1 \,.\, A}{1 \,.\, 2 \ldots a \times 1 \,.\, 2 \ldots a' \times 1 \,.\, 2 \ldots a'' \times 1 \,.\, 2 \ldots a'''}$$

$$\lambda' = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B-1 \,.\, B}{1 \,.\, 2 \ldots b \times 1 \,.\, 2 \ldots b' \times 1 \,.\, 2 \ldots b'' \times 1 \,.\, 2 \ldots b'''}$$

$$\lambda'' = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots C-1 \,.\, C}{1 \,.\, 2 \ldots c \times 1 \,.\, 2 \ldots c' \times 1 \,.\, 2 \ldots c'' \times 1 \,.\, 2 \ldots c'''}$$

2. Then, from the preceding §, λ is the number of ways in which A can be arranged in four divisions of a, a', a'', a''' things, λ' the number of ways in which B things can be arranged in four divisions of b, b', b'', b''' things, and λ'' the number of ways in which C things can be arranged in four divisions of c, c', c'', c'''.

3. But it is evident, that each of these divisions may be combined with each of the other two divisions in all possible ways. Now, since the number of these combinations is equal to the product $\lambda\,\lambda'\,\lambda''$, in like manner the number which arises as often as the numbers A, B, C, agreeably to the proposed conditions, are combined together $=\lambda\,\lambda'\,\lambda''$.

4. What has been here proved for three numbers and four divisions, can, in a similar way, be proved for every other number. If $\therefore$ λ', λ'', λ''', λ^{iv}, λ^{v}, &c. are similar expressions for the numbers B, C, D, E, F, &c. as λ is for the number A, then the required number is always $=\lambda\,\lambda'\,\lambda''\,\lambda'''\,\lambda^{\text{iv}}\,\lambda^{\text{v}}$, &c.

Example. There are 40 balls of four colours, viz. 10 red, 14 blue, 9 green, and 7 white: show

in how many different ways these 40 balls may be arranged in three divisions, so that in the first division there may be 7 red, 5 blue, 3 green, and 2 white; in the second division 2 red, 6 blue, 4 green, and 1 white; and in the third 1 red, 3 blue, 2 green, and 4 white.

Here $A=10$, $B=14$, $C=9$, $D=7$; $a=7$, $b=5$, $c=3$, $d=2$; $a'=2$, $b'=6$, $c'=4$, $d'=1$; $a''=1$, $b''=3$, $c''=2$, $d''=4$; $\therefore$

$$\lambda = \frac{1.2.3.4.5.6.7.8.9.10}{1.2.3.4.5.6.7 \times 1.2 \times 1} = 360$$

$$\lambda' = \frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14}{1.2.3.4.5.6 \times 1.2.3.4.5 \times 1.2.3} = 168168$$

$$\lambda'' = \frac{1.2.3.4.5.6.7.8.9}{1.2.3.4 \times 1.2.3 \times 1.2} = 1260$$

$$\lambda''' = \frac{1.2.3.4.5.6.7}{1.2.3.4 \times 1.2 \times 1} = 105$$

The required number of possible divisions is $\therefore$

$$= \lambda\lambda'\lambda''\lambda''' = 8009505504000.$$

SECTION XXIX.

III.—Auxiliary Rule.

Prob. Let there be more numbers A, B, C, &c. of things of different kinds. It is required to arrange these things in $\mu+\mu'+\mu''+\mu'''+$&c. rows, so that in each of the μ rows there may be a things of the number A, b things of the number B, c things of the number C, and so on; in each of the μ' rows, a' things of the number A, b' things of the number B, c' things of the number C, and so on; in each of the μ'' rows, a'' things of the number A, b'' things of the number B, c'' things of the

number C, and so on. Find the number of all the possible divisions.

Solution 1. If all the rows were different from one another, as, for instance, when they are represented by different numbers, then we could have applied the formulæ of the foregoing § to this case. Then

$$\lambda = \frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots A - 1 . A}{(1 . 2 \ldots a)^{\mu} \times (1 . 2 \ldots a')^{\mu'} \times (1 . 2 \ldots a'')^{\mu''} \times \&c.}$$

$$\lambda' = \frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots B - 1 . B}{(1 . 2 \ldots b)^{\mu} \times (1 . 2 \ldots b')^{\mu'} \times (1 . 2 \ldots b'')^{\mu''} \times \&c.}$$

$$\lambda'' = \frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots C - 1 . C}{(1 . 2 \ldots c)^{\mu} \times (1 . 2 \ldots c')^{\mu'} \times (1 . 2 \ldots c'')^{\mu''} \times \&c.}$$

&c.

and the number of all the possible divisions, as in the preceding §, $= \lambda \lambda' \lambda'' \lambda'''$ &c.

2. But since the problem only requires in general, that the μ rows should be arranged with the combinations of a, b, c, &c. things, the μ' others with the combinations of a', b', c', &c. we must ∴, as we know from the rule of combinations, multiply the number just found by $1 . 2 \ldots \mu \times 1 . 2 \ldots \mu' \times 1 . 2 \ldots \mu'' \times 1 . 2 \ldots \mu'''$ $\times$ &c.

3. The required number of all the possible divisions, according to the conditions of the problem, is ∴

$$= \frac{\lambda \lambda' \lambda'' \lambda''' \ \&c.}{1 . 2 \ldots \mu \times 1 . 2 \ldots \mu' \times 1 . 2 \ldots \mu'' \times \&c.}$$

Remark. Any one of the numbers a, b, c, &c. a', b', c', &c., &c. may be $= 0$. This happens, for instance, when

in a certain row, or in more rows at the same time, one or other of the different kinds of things is altogether wanting. In this case, it is only necessary, for self-evident reasons, to omit from the denominators of λ, λ', λ'', &c. those of the products $1 \, . \, 2 \ldots a$, $1 \, . \, 2 \ldots b$, &c. which refer to the deficient numbers.

SECTION XXX.

PROB. Find the coefficients and the signs of the terms in the solution of $[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$.

Solution 1. If in the numerical expression $[\alpha \beta \gamma \delta \ldots \omega]$ more radical exponents than one are equal to one another, consequently, when this expression assumes the form $[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$, then the terms of the solution have the following general form:

$$(\psi)\ldots\ldots\begin{Bmatrix} \quad (a\alpha \;+ b\beta \;+ c\gamma \;+ \ldots + \;f\zeta) \\ \times \;(a'\alpha + b'\beta + c'\gamma + \ldots + \;l'\lambda) \\ \times \;(a''\alpha + b''\beta + c''\gamma + \;\ldots\; + r''\rho) \\ \times \;\&c. \end{Bmatrix}$$

2. Since in each term of the solution of $[\alpha \beta \gamma \delta \ldots \omega]$, all the letters α, β, γ, $\ldots \omega$ are divided into simple and compound radical exponents (Remark, § 24); then, likewise, in the case when this numerical expression assumes the form $[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$, in each term (ψ)

$$a + a' + a'' + \&c. = \mathfrak{a}, \quad b + b' + b'' + \&c. = \mathfrak{b},$$
$$c + c' + c'' + \&c. = \mathfrak{c}, \; \&c.$$

or, in other words, the radical exponents of the numerical expressions, which occur in each term as factors, are no other than the divisions of $\mathfrak{a}\alpha + \mathfrak{b}\beta + \mathfrak{c}\gamma + \ldots + \mathfrak{k}\kappa$,

as was already observed in § XXVI. We must now, in the first place, find how many terms of the solution of $[\alpha\beta\gamma \ldots\ldots \omega]$ must be combined to form a term of this kind.

3. If we consider those radical exponents in $[\alpha\beta\gamma\ldots\omega]$ which $= \alpha$, as things of one kind, whose number $= \mathfrak{a}$; those which are $= \beta$, as things of a second kind, whose number $= \mathfrak{b}$; those which are $= \gamma$, as things of a third kind, whose number $= \mathfrak{c}$, and so on; the question merely becomes, to find how many terms of the solution of $[\alpha\beta\gamma\ldots\omega]$ are to be combined with the term (ψ); in how many ways the numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, ..., $\mathfrak{k}$ of things of different kinds may be arranged in divisions or rows of the respective numbers $a, b, c, \ldots\ldots f;\ a', b', c', \ldots\ldots l';\ a'', b'', c'', \ldots r''$.

4. Since this is exactly the problem in § XXVIII, when we put $A = \mathfrak{a}$, $B = \mathfrak{b}$, $C = \mathfrak{c}$, &c. we obtain, when N denotes the number of the terms of $[\alpha\beta\gamma\ldots\omega]$, which are to be combined with (ψ)

$$N = \frac{1\,.\,2\,.\,3 \ldots\ldots\ldots\ldots\ldots\ldots \mathfrak{a}-1\,.\,\mathfrak{a}}{1\,.\,2 \ldots a \times 1\,.\,2 \ldots a' \times 1\,.\,2 \ldots a'' \times \&c.} \times$$
$$\frac{1\,.\,2\,.\,3 \ldots\ldots\ldots\ldots\ldots\ldots \mathfrak{b}-1\,.\,\mathfrak{b}}{1\,.\,2 \ldots b \times 1\,.\,2 \ldots b' \times 1\,.\,2 \ldots b'' \times \&c.} \times$$
$$\frac{1\,.\,2\,.\,3 \ldots\ldots\ldots\ldots\ldots\ldots \mathfrak{c}-1\,.\,\mathfrak{c}}{1\,.\,2 \ldots \mathfrak{c} \times 1\,.\,2 \ldots \mathfrak{c}' \times 1\,.\,2 \ldots \mathfrak{c}'' \times \&c.} \times$$
$$\&c.$$

5. Each of the terms which is combined with (ψ), by § XXV. 1 . (b), has the coefficient

$1\,.\,2\ldots m-1 \times 1\,.\,2\ldots m'-1 \times 1\,.\,2\ldots m''-1 \times \ldots\ldots$
when we put $a+b+c+\&c. = m$, $a'+b'+c'+\&c. = m'$, $a''+b''+c''+\&c. = m''$, and so on.

6. By § XVI., when $[\alpha\beta\gamma\ldots\omega]$ is transformed into $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\kappa^{\mathfrak{k}}]$, the latter expression must have a coefficient κ, by which ∴ each term of the development, consequently also the term (ψ), must be divided. But by the same §

$$\kappa = 1\,.\,2\ldots\mathfrak{a} \times 1\,.\,2\ldots\mathfrak{b} \times 1\,.\,2\ldots\mathfrak{c} \times \ldots$$

7. From 4, 5, 6, it follows, when the coefficient of the term (ψ) is represented by K, that

$$K = \frac{1.2\ldots m-1 \times 1.2\ldots m'-1 \times 1.2\ldots m''-1 \times \&c.}{1.2\ \ldots\ldots\ \mathfrak{a} \times 1.2\ldots\ldots\ldots\mathfrak{b} \times 1.2\ldots\ldots\ldots\mathfrak{c} \times \&c.}\,.\ N$$

or if for N its value in 4 be substituted

$$K = \frac{1\,.\,2\ldots m-1 \times 1\,.\,2\ldots m'-1 \times 1\,.\,2\ldots m''-1 \times \&c.}{\left[\begin{array}{c} 1\,.\,2\ldots a \times 1\,.\,2\ldots b \times 1\,.\,2\ldots c \times \&c. \\ \times\ 1\,.\,2\ldots a' \times 1\,.\,2\ldots b' \times 1\,.\,2\ldots c' \times \&c. \\ \times\ 1\,.\,2\ldots a'' \times 1\,.\,2\ldots b'' \times 1\,.\,2\ldots c'' \times \&c. \\ \&c. \end{array}\right]}\,.$$

8. Hitherto it has been assumed, that all the numerical expressions which occur in the term (ψ) as factors, are different from one another. If this be not the case, or if the term have the form

$$\begin{array}{l} (a\alpha + b\beta + c\gamma + \ldots\ldots + f\zeta)^{\mu} \\ +\ (a'\alpha + b'\beta + c'\gamma + \ldots\ldots + l'\lambda)^{\mu'} \\ +\ (a''\alpha + b''\beta + c''\gamma + \ldots\ldots + r''\rho)^{\mu''} \\ +\ \&c. \end{array}$$

then the number of divisions (§ XXIX.) which the numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, &c. admit of, is

$$N = \frac{1}{1\,.\,2\ldots\mu\,\times\,1\,.\,2\ldots\mu'\,\times\,1\,.\,2\ldots\mu''\times\&c.}\times$$
$$\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\,\mathfrak{a}-1\,.\,\mathfrak{a}}{(1\,.\,2\ldots a)^{\mu}\times(1\,.\,2\ldots a')^{\mu'}\times(1\,.\,2\ldots a'')^{\mu''}\times\&c.}\times$$
$$\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\,\mathfrak{b}-1\,.\,\mathfrak{b}}{(1\,.\,2\ldots b)^{\mu}\times(1\,.\,2\ldots b')^{\mu'}\times(1\,.\,2\ldots b'')^{\mu''}\times\&c.}\times$$
$$\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\,\mathfrak{b}-1\,.\,\mathfrak{c}}{(1\,.\,2\ldots c)^{\mu}\times(1\,.\,2\ldots c')^{\mu'}\times(1\,.\,2\ldots c'')^{\mu''}+\&c.}\times$$
$$\&c.$$

9. But from § XXV. 1 . (b), the coefficient of every term which is combined with the term (ψ)

$$= (1\,.\,2\ldots m-1)^{\mu}\times(1\,.\,2\ldots m'-1)^{\mu}\times$$
$$(1\,.\,2\ldots m''-1)^{\mu''}\times\ldots$$

Further, as in § VI.

$$\kappa = 1\,.\,2\ldots\mathfrak{a}\times 1\,.\,2\ldots\mathfrak{b}\times 1\,.\,2\ldots\mathfrak{c}\times\ldots$$

10. We ∴ have for this case

$$K = \frac{\left[\begin{array}{c}(1\,.\,2\ldots m-1)^{\mu}\times(1\,.\,2\ldots m'-1)^{\mu'}\times\\ (1\,.\,2\ldots m''-1)^{\mu''}\times\&c.\end{array}\right]}{1\,.\,2\ldots\mathfrak{a}\times 1\,.\,2\ldots\mathfrak{b}\times 1\,.\,2\ldots\mathfrak{c}\times\&c.}\,.\,N$$

or, when for N its value in 8 is substituted,

$$K = \frac{\left[\begin{array}{c}(1\,.\,2\ldots m-1)^{\mu}+(1\,.\,2\ldots m'-1)^{\mu'}\times\\ (1\,.\,2\ldots m''-1)^{\mu''}\times\&c.\end{array}\right]}{\left(\begin{array}{l}1\,.\,2\ldots\mu\times 1\,.\,2\ldots\mu'\times 1\,.\,2\ldots\mu''\times\quad\&c.\\ \times\,(1\,.\,2\ldots a)^{\mu}\times(1\,.\,2\ldots a')^{\mu'}\times(1\,.\,2\ldots a'')^{\mu''}\times\&c.\\ \times\,(1\,.\,2\ldots b)^{\mu}\times(1\,.\,2\ldots b')^{\mu'}\times(1\,.\,2\ldots b'')^{\mu''}\times\&c.\\ \times\,(1\,.\,2\ldots c)^{\mu}\times(1\,.\,2\ldots c')^{\mu'}\times(1\,.\,2\ldots c'')^{\mu''}\times\&c.\\ \times\,\ldots\ldots\ldots\ldots\,\&c.\end{array}\right)}$$

or

11.

$$K = \frac{\left[\begin{array}{c}(1 \,.\, 2 \ldots m - 1)^{\mu} \times (1 \,.\, 2 \ldots m' - 1)^{\mu'} \times \\ (1 \,.\, 2 \ldots m'' - 1)^{\mu''} \times \&c.\end{array}\right]}{\left[\begin{array}{l}1 \,.\, 2 \ldots {}^{\mu} \times 1 \,.\, 2 \ldots {}^{\mu'} \times 1 \,.\, 2 \ldots {}^{\mu''} \times \&c. \\ \times\ (1 \,.\, 2 \ldots a \times 1 \,.\, 2 \ldots b \times 1 \,.\, 2 \ldots c \times \&c.)^{\mu} \\ \times\ (1 \,.\, 2 \ldots a' \times 1 \,.\, 2 \ldots b' \times 1 \,.\, 2 \ldots c' \times \&c.)^{\mu'} \\ \times\ (1 \,.\, 2 \ldots a'' \times 1 \,.\, 2 \ldots b'' \times 1 \,.\, 2 \ldots c'' \times \&c.)^{\mu''} \\ \times\ \ldots\ldots\ldots \ \&c.\end{array}\right]}$$

12. This expression for K includes that in 7, because we obtain the former when we substitute $\mu = \mu' = \mu'' = \&c. = 1$ in the latter; it is consequently quite general, and obtains for all imaginable cases.

In the case where one of the numbers m, m', m'', &c. say $m, = 1$, we must, instead of the product $1 \,.\, 2 \ldots m - 1$, merely put 1.

13. The sign of the term (ψ) is always the same as that of the expression $(\overline{+m})^{\mu'} \times (\overline{+m'})^{\mu'} \times (\overline{m''})^{\mu''}$, when every time we merely give the numbers m, m', m'', &c., the sign $-$ when even, and the sign $+$ when odd. The reason of this follows from § XXV., 1. (c.)

Example. Determine the coefficient and the sign of the term

$$(3a + 7\beta \times 2\gamma + 4\delta)\ (5a + 4\gamma + \delta)^3\ (5a)^4$$

of the solution of $(a^{38} \beta^7 \gamma^{14} \delta^7)$.

Here $a = 3$, $b = 7$, $c = 2$, $d = 4$; $a' = 5$, $b' = 0$, $c' = 4$, $d' = 1$; $a'' = 5$; further $\mu = 1$, $\mu' = 3$, $\mu'' = 4$; $\therefore m = a + b + c + d = 16$, $m' = a' + b' + c' + d' = 10$, $m'' = a'' = 5$. We have consequently from the formula in 11.

$$K = \frac{(1\,.\,2\,.\,3 \ldots 15)^1 \times (1\,.\,2\,.\,3 \ldots 9)^3 \times (1\,.\,2\,.\,3\,.\,4)^4}{\left[\begin{array}{l} \quad 1 \times 1\,.\,2\,.\,3 \times 1\,.\,2\,.\,3\,.\,4 \\ \times\ (1\,.\,2\,.\,3 \times 1\,.\,2 \ldots 7 \times 1\,.\,2 \times 1\,.\,2\,.\,3\,.\,4)^1 \\ \times\ (1\,.\,2\,.\,3\,.\,4\,.\,5 \times 1\,.\,2\,.\,3\,.\,4 \times 1)^3 \\ \times\ (1\,.\,2\,.\,3\,.\,4\,.\,5)^4 \end{array}\right]}$$

$$= \frac{500594094}{25}.$$

The sign of the term is the same as that of the expression $(-16)^1 \times (-10)^3 \times (+5)^4$, consequently $+$.

SECTION XXXI.

In order to render what has been already advanced more intelligible, I shall here give the complete solution of the numerical expression $(a^3\beta^3)$, which has already been made use of in § XXVI. as an example for the representation of involution. The terms are arranged as they are there found.

$$
\begin{aligned}
(a^3\beta^3) = {} & \\
& \tfrac{1}{36}(a)^3(\beta)^3 - \tfrac{1}{12}(a)(2a)(\beta)^3 + \tfrac{1}{18}(3a)(\beta)^3 \\
& - \tfrac{1}{4}(a)^2(\beta)^2(a+\beta) + \tfrac{1}{4}(2a)(\beta)^2(a+\beta) + \tfrac{1}{1}(a)(\beta)^2(2a+\beta) \\
& - \tfrac{1}{2}(\beta)^2(3^a+\beta) - \tfrac{1}{12}(a)^3(\beta)(2\beta) + \tfrac{1}{4}(a)(2a)(\beta)(2\beta) \\
& - \tfrac{1}{6}(3a)(\beta)(2\beta) + \tfrac{1}{2}(a)^2(\beta)(a+2\beta) + \tfrac{1}{2}(a)(\beta)(a+\beta)^2 \\
& - \tfrac{1}{2}(2a)(\beta)(a+2\beta) - (\beta)(a+\beta)(2a+\beta) - \tfrac{3}{2}(a)(\beta)(2a+2\beta) \\
& + 2(\beta)(3a+2\beta) + \tfrac{1}{4}(a)^2(2\beta)(a+\beta) - \tfrac{1}{4}(2a)(2\beta)(a+\beta) \\
& - \tfrac{1}{2}(a)(2\beta)(2a+\beta) + \tfrac{1}{2}(2\beta)(3a+\beta) + \tfrac{1}{18}(a)^3(3\beta) \\
& - \tfrac{1}{6}(a)(2a)(3\beta) + \tfrac{1}{3}(3a)(3\beta) - \tfrac{1}{2}(a)^2(a+3\beta) \\
& - (a)(a+\beta)(a+2\beta) - \tfrac{1}{6}(a+\beta)^3 + \tfrac{1}{2}(2a)(a+3\beta) \\
& + (a+2\beta)(2a+\beta) + \tfrac{1}{2}(a+\beta)(2a+2\beta) + 2(a)(2a+3\beta) \\
& - \tfrac{10}{3}(3a+3\beta).
\end{aligned}
$$

From this example we shall clearly perceive, how we are to proceed in every other case, and it seems to me unnecessary to add any thing more.

III.—ON THE VALUES OF THE FUNCTIONS OF THE ROOTS OF AN EQUATION WHICH ARE NOT SYMMETRICAL, AND ON THE METHOD BY WHICH TO MAKE THESE VALUES DEPENDANT ON EQUATIONS.

SECTION XXXII.

SYMMETRICAL functions differ from the others in this, that in the first place they contain all the roots of the given equation; and secondly, these roots are so combined with one another, that the functions in each transposition are changed. For this kind of functions, it is sufficient only to mention in general the form of the combination, without referring to the roots themselves, because we are always sure before-hand to obtain the same results in the composition. Thus, for instance, the expression (12) is fully determined, although by the notation nothing more is indicated, than that we are to take the sum of all the products which arise from the combination of every root with the square of another. A function of this kind ∴ can only have a single value, and this value is no other than that, the way to find which, was shown in the foregoing §. It is, as we have already seen, in reference to the coefficients of the given equation, always rational, and it must necessarily be so, because otherwise the function would have more values.

But this is not the case with the other functions. If

we merely wished to give the form of their combination, we should not by this means alone be able to determine the function. If, for instance, a, b, c, be the three roots of an equation of the third degree, then indeed the sum of all the three roots $a+b+c$, can only be expressed in one way; on the other hand, the sum of two roots, in three different ways, viz. by $a+b$, $a+c$, $b+c$; and the difference of two roots may be expressed in as many as six different ways, viz. by $a-b$, $b-a$, $a-c$, $c-a$, $b-c$, $c-b$. A function of this kind consequently has more values, which arise partly from the substitution of the roots it contains for the remaining ones, and partly from the transposition of these roots. None of these, so long as the roots a, b, c, &c. are undetermined, can be found by themselves without the others, because otherwise there would be no reason why we should exactly find this or that value, and not the others likewise. Hence it follows, that the value of every function which is not symmetrical, can only be expressed by an equation which, at the same time, includes all the values which the function can contain by substitution and transposition of the roots.

Whatever may seem obscure in these general observations, the following problems will render clear. I here remind my readers, once for all, that I shall not in future denote the roots of the given equation, as heretofore, by a, b, c, d, &c., but by x', x'', x''', x^{IV}, &c. This I do, partly, because this notation has been adopted by nearly the whole of the modern analysts, and because it is always desirable to retain the mode of notation already adopted, if it can be done without disadvantage; partly

also, because we are accustomed to associate with the first letters of the alphabet, the idea of determinate numerical values; while, on the contrary, here for instance, x' denotes neither this nor that determinate root, but generally any root whatever, and the dashes over the x are put for the sake of distinction.

SECTION XXXIII.

Prob. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

determine the value of the function $x'x''$, without knowing the roots of the equation.

Solution **1.** Since x', x'', x''', are the roots of the given equation, we have

$$\begin{aligned} x' + x'' + x''' &= A \\ x'x'' + x'x''' + x''x''' &= B \\ x'x''x''' &= C \end{aligned}$$

and from these three equations we must endeavour to determine the value of $x'x''$.

2. With this view, put $x'x'' = t$, substitute this value in the second and third equations, and multiply the first equation by x'''; this gives

$$\begin{aligned} x'x''' + x''x''' + x'''^2 &= Ax''' \\ t + x'x''' + x''x''' &= B \\ x'''t \qquad\qquad &= C \end{aligned}$$

If we subtract the first of these equations from the second, and then substitute for x''' its value $\frac{C}{t}$ from the third equation, we obtain the following equations for t:

$$t^3 - Bt^2 + ACt - C^2 = 0$$

The value of $x'x''$ can only be found by the solution of an equation of the third degree.

3. If we had put $x'x'''$, or $x''x''' = t$, we should have found the same equation. Hence we may safely conclude, that the values $x'x''$, $x'x'''$, $x''x'''$, must be the three roots of the equation found.

4. This consequence might very easily have been foreseen. Since, then, there is no reason why the equation for t should give exactly the product $x'x''$, and not also the product $x'x'''$, or $x''x'''$, as the three roots x', x'', x''', are in some way contained in the given equation, consequently it must necessarily be of the third degree.

5. We could ∴ also have found this equation in a direct way. Since, for instance, $x'x''$, $x'x'''$, $x''x'''$, must be the three roots of the required equation, consequently it can be no other than the following equation:

$$(t - x'x'')\,(t - x'x''')\,(t - x''x''') = 0.$$

If we actually multiply the three factors in the first part, we obtain

$$\begin{aligned} t^3 &- (x'x'' + x'x''' + x''x''')\,t^2 \\ &+ (x'^2x''x''' + x'x''^2x''' + x'x''x'''^2)t \\ &- x'^2x''^2x'''^2 = 0 \end{aligned}$$

or since $x'x'' + x'x''' + x''x'' = B$, $x'^2x''x''' + x'x''^2x''' + x'x''x'''^2 = (x' + x'' + x''')\,x'x''x''' = AC$, $x'^2x''^2x'''^2 = (x'x''x''')^2 = C^2$, the same equation as we have above.

Example. In the equation $x^3 + x^2 - 32\,x - 60 = 0$, $A = -1$, $B = -32$; $C = 60$; ∴ we have

$$t^3 + 32t^2 - 60t - 3600 = o,$$

an equation, whose roots are the product of every two roots of the former equation. Thus the roots of this equation are $+ 10, - 12, - 30$, and the roots of the former are $- 2, - 5, + 6$.

PROB. From the given equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = o$$

determine the value of the function x'^2.

Solution 1. Since all the roots are contained in the given equation in the same way, and for the root whose square is required, nothing nearer can be determined; consequently the square of one of the roots cannot be found, without at the same time finding the squares of all the remaining ones. Therefore the equation by which x'^2 is expressed, must necessarily be of the fourth degree.

2. If $\therefore$ we put $x'^2 = t$, then the equation which gives the value of t must be of the fourth degree, and its roots must be x'^2, x''^2, x'''^2 x^{IV2}. This equation is $\therefore$ composed of the four particular equations

$$t - x'^2 = o,\ t - x''^2 = o$$
$$t - x'''^2 = o,\ t - x^{IV2} = o$$

and consequently is no other than the product of these last. By actual multiplication we have

$$t^4 - (x'^2 + x''^2 + x'''^2 + x^{IV2})\, t^3 +$$
$$(x'^2x''^2 + x'^2x'''^2 + x'^2x^{IV2} + x''^2x'''^2 + x''^2x^{IV2} + x'''^2x^{IV2})t^2$$
$$- (x'^2x''^2x'''^2 + x'^2x''^2x^{IV2} + x'^2x'''^2x^{IV2} + x''^2x'''^2x^{IV2})\, t$$
$$+ x'^2x''^2x'''^2x^{IV2} = o$$

It only remains now to express the coefficients of this equation by the coefficients of the given one.

3. But this equation, when we use brackets, may be thus represented:

$$t^4 - (2)\,t^3 + (2^2)\,t^2 - (2^3)\,t + (2^4) = 0$$

and the values of the numerical expressions are obtained directly from the annexed Tables. Thus, since the coefficients E, F, &c. $= 0$, we find

$$(2) = A^2 - 2B, \quad (2^2) = B^2 - 2AC + 2D,$$
$$(2^3) = C^2 - 2BD, \; (2^4) = D^2,$$

By substituting these values, we have

$$t^4 - (A^2 - 2B)\,t^3 + (B^2 - 2AC + 2D)\,t^2 - (C^2 - 2BD)\,t + D^2 = 0$$

which is the equation sought.

4. We could also have found this equation in the following way. Put $x^2 = t$, or $x = \sqrt{t}$, substitute this value of x in the given equation, and place all the terms in which $\sqrt{t}$ is involved on one side of the equation. Hence we obtain

$$t^2 + Bt + D = (At \mp C)\sqrt{t}.$$

If this equation be squared, and properly arranged, we obtain the same equation as in 3.

Example. In the equation $x^4 + 10x^3 + 25x^2 - 2x - 12 = 0$, we have $A = -10$, $B = 25$, $C = 2$, $D = -12$. The equation, whose roots are the squares of the roots of this equation, is $\therefore$

$$t^4 - 50\,t^3 + 641\,t^2 - 604\,t + 144 = 0.$$

The roots of the former equation are $-3+\sqrt{5}$, $-3-\sqrt{5}$, $-2+\sqrt{7}$, $-2-\sqrt{7}$; the roots of the latter are $14-6\sqrt{5}$, $14+6\sqrt{5}$, $11-4\sqrt{7}$, $11+4\sqrt{7}$; these last are the squares of the former.

SECTION XXXV.

Prob. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

find the value of the function $ax'x''+bx'''$.

Solution 1. The equation, by which the function $ax'x''+bx'''$ is expressed, or on which it depends, must contain all its possible values. Since in this function all the three roots occur, it is only necessary, in order to find its different values, to transpose these roots in all possible ways, and retain those of the results which differ from one another. But the results of these transpositions are

$$ax'x''+bx''',\ ax'x'''+bx'',\ ax''x'''+bx'$$
$$ax''x'+bx''',\ ax'''x'+bx'',\ ax'''x''+bx'$$

amongst which there are only three which are different from one another, viz.

$$ax'x''+bx''',\ ax'x'''+bx'',\ ax''x'''+bx'.$$

The required equation must consequently have these three functions for roots.

2. If ∴ we denote each of these undetermined functions by t, we then obtain the three particular equations.

$$t-(ax'x''+bx''')=0$$
$$t-(ax'x'''+bx'')=0$$
$$t-(ax''x'''+bx')=0$$

and the product of these gives the equation, from which the value of each of the three functions must be determined. It may be represented by

$$t^3 - A't^2 + B't - C' = 0.$$

3. Then

$$A' = (ax'x'' + bx''') + (ax'x''' + bx'') + (ax''x''' + bx')$$
$$= a(1^2) + b(1)$$
$$B' = (ax'x'' + bx''')(ax'x''' + bx'')$$
$$+ (ax'x'' + bx''')(ax''x''' + bx')$$
$$+ (ax'x''' + bx'')(ax''x''' + bx')$$
$$= a^2(1^2 2) + ab(1\ 2) + b^2(1^2)$$
$$C' = (ax'x'' + bx''')(ax'x''' + bx'')(ax''x''' + bx')$$
$$= a^3(2^3) + a^2b(1^2 3) + ab^2(2^2) + b^3(1^3)$$

If for the numerical expressions we substitute their values from the annexed Tables, we then obtain, since D, E, &c. $= 0$,

$$A' = aB + bA$$
$$B' = a^2AC + ab(AB - 3C) + b^2B$$
$$C' = a^3C^2 + a^2b(A^2C - 2BC) + ab^2(B^2 - 2AC) + b^3C,$$

and consequently the equation on which the required function depends, is

$$t^3 - (aB + bA)t^2 + [a^2AC + ab(AB - 3C) + b^2B]t$$
$$-[a^3C^2 + a^2b(A^2C - 2BC) + ab^2(B^2 - 2AC) + b^3C] = 0$$

SECTION XXXVI.

Prob. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

find the equation on which the function $x' - x''$ depends.

Solution. The function $x' - x''$ contains the six follow-

ing values, which arise partly from the substitution, and partly from the transposition of the roots:

$$x'-x'',\ x'-x''',\ x''-x'''$$
$$x''-x',\ x'''-x',\ x'''-x''.$$

The required equation is $\therefore$ the product of the following six particular equations:

$$t-(x'-x'')=0,\ t+(x'-x'')=0$$
$$t-(x'-x''')=0,\ t+(x'-x''')=0$$
$$t-(x''-x''')=0,\ t+(x''-x''')=0$$

or, when every two of the opposite equations are multiplied together, the product of the three following equations:

$$t^2-(x'-x'')^2=0$$
$$t^2-(x'-x''')^2=0$$
$$t^2-(x''-x''')^2=0$$

from which it follows, that it only contains even powers of t. Let $\therefore$

$$t^6-A't^4+B't^2-C'=0$$

be this equation; then

$$A'=(x'-x'')^2+(x'-x''')^2+(x''-x''')^2$$
$$=2(2)-2(1^2)$$
$$B'=(x'-x'')^2(x'-x''')^2+(x'-x'')^2(x''-x''')^2$$
$$+(x'-x''')^2(x''-x''')^2$$
$$=(4)+3(2^2)-2(13)$$
$$C'=(x'-x'')^2(x'-x''')^2(x''-x''')^2$$
$$=(24)-2(3^2)-6(2^3)+2(123)-2(1^24)$$

Now, if we take the values of the numerical expressions from the Tables, after the proper reduction, we find

$$A'=2A^2-6B$$
$$B'=A^4-6A^2B+9B^2$$
$$C'=A^2B^2-4B^3-4A^3C+18ABC-27C^2$$

and $\therefore$ the required equation is

$$t^6 - (2A^2 - 6B)\,t^4 + (A^4 - 6A^2B + 9B^2)\,t^2$$
$$- (A^2B^2 - 4B^3 - 4A^3C + 18ABC - 27C^2) = 0,$$

whose roots are the differences of the roots of the given equation.

Example. In the equation $x^3 - 8x^2 + 19x - 12 = 0$ $A = 18$, $B = 19$, $C = 12$; $\therefore A' = 14$, $B' = 49$, $C' = 36$, and consequently the equation for the differences is

$$t^6 - 14\,t^4 + 49\,t^2 - 36 = 0.$$

The roots of the first equation are $+1$, $+3$, $+4$; the roots of the last are $+1$, -1, $+2$, -2, $+3$, -3, as required.

SECTION XXXVII.

Prob. From the given equation

$$t^3 - Ax^2 + Bx - C = 0,$$

find the equation for the fraction $\frac{x'}{x''}$.

Solution 1. The function $\frac{x'}{x''}$, by the substitution and transposition of the roots, contains the six following values:

$$\frac{x'}{x''},\ \frac{x''}{x'},\ \frac{x'}{x'''},\ \frac{x'''}{x'},\ \frac{x''}{x'''},\ \frac{x'''}{x''}.$$

The equation by which these functions are expressed, is $\therefore$ of the sixth degree.

2. This equation is represented by

$$t^6 - A't^5 + B't^4 - C't^3 + D't^2 - E't + F' = 0,$$

consequently A' is the sum of the functions given in 1, B' the sum of their products, two and two, C' the sum of

their products three and three, and so on. Hence, after the usual reduction, we get the following values:

$$A' = \frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$= \frac{(12)}{(1^3)}$$

$$B' = 3 + \frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$+ \frac{x'^3x''^3 + x'^3x'''^3 + x''^3x'''^3 + x'x''x'''^4 + x'x''^4x''' + x'^4x''x'''}{x'^2x''^2x'''^2}$$

$$= 3 + \frac{(12)}{(1^3)} + \frac{(3^2) + (1^24)}{(1^3)^2}$$

$$C' = 2 + 2.\frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$+ \frac{x'^2x''^4 + x'^4x''^2 + x'^2x'''^4 + x'^4x'''^2 + x''^2x'''^4 + x''^4x'''^2}{x'^2x''^2x'''^2}$$

$$= 2 + 2.\frac{(12)}{(1^3)} + \frac{(24)}{(1^3)^2}$$

If we proceed further with the calculation, we find the same expression for D' as for B', and for E' the same expression as for A'; further, F', as the product of all the above six functions, $= 1$.

3. Now, if we substitute for the numerical expressions their values taken from the Tables, we obtain

$$A' = E' = \frac{AB - 3C}{C}$$

$$B' = D' = \frac{B^3 - 5ABC + A^3C + 6C^2}{C^2}$$

$$C' = \frac{6ABC - 7C^2 + A^2B^2 - 2B^3 - 2A^3C}{C^2}$$

$$F' = 1$$

and these are the coefficients of the assumed equation for t.

Example. In the equation $x^3 + 2x - x - 2 = 0$, we have $A = -2$, $B = -1$, $C = 2$. Hence we find $A' = E' = -2$, $B' = D' - \frac{13}{4}$, $C' = \frac{17}{2}$. Consequently the required equation is

$$t^6 + 2t^5 - \frac{13}{4}t^4 - \frac{17}{2}t^3 - \frac{13}{4}t^2 + 2t + 1 = 0$$

The roots of this equation are -1, -1, -2, $-\frac{1}{2}$, $+2$, $+\frac{1}{2}$, which must be the case, since $+1$, -1, -2 are the roots of the given equation.

SECTION XXXVIII.

The equations for t, which we found in the foregoing §, and which may be found in a similar way for all other functions, may, in reference to those which are given, be called transformed equations. The degree and form of these last depend upon the functions which we assume for t. In the functions of which we have hitherto treated, the transformed equation has always been either of a higher or of the same degree as the given one. But there are functions for which the given equation is of a lower degree; and in this case it can sometimes serve to solve the given equation, as will be shown by the two following examples for equations of the fourth degree.

SECTION XXXIX.

Prob. From the given equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

find the value of the function $x'x'' + x'''x^{iv}$.

Solution 1. Since in the function $x'x'' + x'''x^{iv}$ all the roots occur at once, they can only alter their value by transposition. But the four roots x', x'', x''', x^{iv}, may be transposed in 1 . 2 . 3 . 4 = 24 ways, and the results thus obtained are

$$\begin{array}{lll}
x'x'' + x'''x^{iv}, & x'x''' + x''x^{iv}, & x'x^{iv} + x''x''' \\
x'x'' + x^{iv}x''', & x'x''' + x^{iv}x'', & x'x^{iv} + x'''x'' \\
x''x' + x'''x^{iv}, & x'''x' + x''x^{iv}, & x^{iv}x' + x''x''' \\
x''x' + x^{iv}x''', & x'''x' + x^{iv}x'', & x^{iv}x' + x'''x'' \\
x'''x^{iv} + x'x'', & x''x^{iv} + x'x''', & x''x''' + x'x^{iv} \\
x^{iv}x''' + x'x'', & x^{iv}x'' + x'x''', & x'''x'' + x'x^{iv} \\
x'''x^{iv} + x''x', & x''x^{iv} + x'''x', & x''x''' + x^{iv}x' \\
x^{iv}x''' + x''x', & x^{iv}x'' + x'''x', & x'''x'' + x^{iv}x'
\end{array}$$

It is immediately seen, that eight of these results, which are in the same vertical column, are always equal to one another, and that also the function can have no more than the three following different values:

$$x'x'' + x'''x^{iv},\ x'x''' + x''x^{iv},\ x'x^{iv} + x''x'''$$

The transformed equation is consequently of the third degree, and its roots are the values just mentioned.

2. Let this equation be expressed by

$$t^3 - A't^2 + B't - C' = 0;$$

then, from the nature of equations,

$$A' = (x'x'' + x'''x^{\prime v}) + (x'x''' + x''x^{\prime v}) + (x'x^{\prime v} + x''x''')$$
$$= (1^2)$$
$$B' = (x'x'' + x'''x^{\prime v})\,(x'x''' + x''x^{\prime v})$$
$$+ \,(x'x'' + x'''x^{\prime v})\,(x'x^{\prime v} + x''x''')$$
$$+ \,(x'x''' + x''x^{\prime v})\,(x'x^{\prime v} + x''x''')$$
$$= (1^2 2)$$
$$C' = (x'x'' + x'''x^{\prime v})\,(x'x''' + x''x^{\prime v})\,(x'x^{\prime v} + x''x''')$$
$$= (2^3) + (1^3 3)$$

3. Now if we take the values of the numerical expressions from the annexed Tables, and then substitute for A', B', C', their values found in the assumed equation, we obtain the required transformed equation

$$t^3 - Bt^2 + (AC - 4D)\,t - (C^2 - 4BD + A^2D) = 0$$

I shall now proceed to show what use can be made of this equation in the general solution of equations of the fourth degree.

SECTION XI.

Let it be assumed, that we can find a root of the transformed equation; let t' be this root, $\therefore\ x'x'' + x'''x^{\prime v} = t'$. It only remains now, from this equation, together with the others, which express the known relations between the roots and the coefficients, to determine the values of x', x'', x''', $x^{\prime v}$.

For this purpose, combine, first, the two equations

$$x'x'' + x'''x^{\prime v} = t',\ x'x''x'''x^{\prime v} = D.$$

These give

$$(x'x'' - x'''x^{\prime v})^2 = (x'x'' + x'''x^{\prime v})^2 - 4x'x''x'''x^{\prime v}$$
$$= t'^2 - 4D$$
$$x'x'' - x'''x^{\prime v} = \sqrt{(t'^2 - 4D)}$$

$\therefore$

$$x'x''=\frac{t'+\sqrt{(t'^2-4D)}}{2},\ x'''x^{\text{IV}}=\frac{t'-\sqrt{(t'^2-4D)}}{2}$$

Combine the two equations

$$x'''x^{\text{IV}}(x'+x'')+x'x''(x'''+x^{\text{IV}})=B$$
$$(x'+x'')+\qquad(x'''+x^{\text{IV}})=A$$

These give

$$x'+x''=\frac{Ax'x''-B}{x'x''-x'''x^{\text{IV}}}=\frac{At'-2B}{2\sqrt{(t'^2-4D)}}+\frac{A}{2}$$
$$x'''+x^{\text{IV}}=\frac{Ax'''x^{\text{IV}}-B}{x'''x^{\text{IV}}-x'x''}=\frac{At'-2B}{2\sqrt{(t'^2-4D)}}+\frac{A}{2}$$

Now we know the values of $x'x''$, $x'+x''$, $x'''x^{\text{IV}}$, $x'''+x^{\text{IV}}$. From the two first of these values we can determine the roots x', x'', and from the two last the roots x''', x^{IV}, merely by the solution of quadratic equations.

In this solution it is only sufficient to know one root of the transformed equation.

SECTION XLI.

Prob. From the given equation of the fourth degree

$$x^4-Ax^3+Bx^2-Cx+D=0,$$

determine the value of the function $(x'+x''-x'''-x^{\text{IV}})^2$.

Solution 1. If we proceed with this function, as we have already done with the function $x'x''+x'''x^{\text{IV}}$, we shall then find no more than the three following different values:

$$(x'+x''-x'''-x^{\text{IV}})^2,\qquad(x'+x'''-x''-x^{\text{IV}})^2$$
$$(x'+x^{\text{IV}}-x''-x''')^2$$

which consequently likewise depend on an equation of the third degree. This equation we could find in the usual

way: the following method, however, in this case, leads much more readily to the desired object.

2. Thus if we put $(x' + x'' - x''' - x^{IV})^2 = t$; then

$$t = x'^2 + x''^2 + x'''^2 + x^{IV2} + 2x'x'' - 2x'x''' - 2x'x^{IV} - 2x''x''' - 2x''x^{IV} + 2x'''x^{IV}$$

$$= (x' + x'' + x''' + x^{IV})^2$$
$$- 4(x'x'' + x'x''' + x'x^{IV} + x''x''' + x''x^{IV} + x'''x^{IV})$$
$$+ 4(x'x'' + x'''x^{IV})$$
$$= [1]^2 - 4[1^2] + (4x'x'' + x'''x^{IV})$$
$$= A^2 - 4B + 4(x'x'' + x'''x^{IV})$$

and $\therefore$

$$x'x'' + x'''x^{IV} = \frac{t - A^2 + 4B}{4}$$

3. Now since $x'x'' + x'''x^{IV}$ is exactly the function for which in § XXXIX the equation
$t^3 - Bt^2 + (AC - 4D)t - (C^2 - 4BD + A^2D) = 0$
was found, consequently in this equation it is only necessary to put $\frac{t - A^2 + 4B}{4}$ for t. Hence we find the equation

$$t^3 - (3A^2 - 8B)t^2 + (3A^4 - 16A^2B + 16B^2 + 16AC - 64D)t - (A^3 - 4AB + 8C)^2 = 0,$$

whose roots are the functions given in 1.

Hence also we may obtain, as in the preceding §, a solution of equations of the fourth degree, which may be seen immediately; only it is here assumed, that all the three roots of this equation are already found.

SECTION XLII.

Let t', t'', t''', be the three roots of the transformed equation in the foregoing §, then we have

$$(x' + x'' - x''' - x'^{v})^2 = t'$$
$$(x' + x''' - x'' - x'^{v})^2 = t''$$
$$(x' + x'^{v} - x'' - x''')^2 = t'''$$

and consequently

$$x' + x'' - x''' - x'^{v} = \sqrt{t'}$$
$$x' + x''' - x'' - x'^{v} = \sqrt{t''}$$
$$x' + x'^{v} - x' - x''' = \sqrt{t'''}$$

If we combine these three equations with

$$x' + x'' + x''' + x'^{v} = A$$

we obtain, merely by addition and subtraction, the following expressions for the roots:

$$x' = \frac{A + \sqrt{t'} + \sqrt{t''} + \sqrt{t'''}}{4}$$

$$x'' = \frac{A + \sqrt{t'} - \sqrt{t''} - \sqrt{t'''}}{4}$$

$$x''' = \frac{A - \sqrt{t'} + \sqrt{t''} - \sqrt{t'''}}{4}$$

$$x'^{v} = \frac{A - \sqrt{t'} - \sqrt{t''} + \sqrt{t'''}}{4}$$

Instead of which .·. we have, so soon as we have solved the expression for t, the four roots of the given equation.

But here there appears an evident difficulty. Thus, since in the values of x', x'', x''', x'^{v}, there are three roots $\sqrt{t'}$, $\sqrt{t''}$, $\sqrt{t'''}$, and these roots may be assumed to be either positive or negative,—the question is, how we are to proceed in order to determine the signs. For this purpose consider the last term of the transformed equation. Since this term must be the product of its three roots, we have

$$(A^3 - 4AB + 8C)^2 = t't''t''';$$

$\therefore$

$$A^3-4AB+8C=\sqrt{t't''t'''}=\sqrt{t'}.\sqrt{t''}.\sqrt{t'''}.$$

Now if $A^3-4AB+8C$ be positive, then also the product $\sqrt{t'}\,.\,\sqrt{t''}\,.\,\sqrt{t'''}$ must be positive, and consequently the signs can only be combined together in the four following ways:

$$\begin{array}{lll} +\sqrt{t'}, & +\sqrt{t''}, & +\sqrt{t'''} \\ +\sqrt{t'}, & -\sqrt{t''}, & -\sqrt{t'''} \\ -\sqrt{t'}, & +\sqrt{t''}, & -\sqrt{t'''} \\ -\sqrt{t'}, & -\sqrt{t''}, & +\sqrt{t'''} \end{array}$$

and these combinations give the above values for x', x'', x''', x^{IV}.

If, on the other hand, $A^3-4AB+8C$ be negative, then the product $\sqrt{t'}\,.\,\sqrt{t''}\,.\,+t'''$ is also negative, and consequently in this case the signs can only be combined in the four following ways:

$$\begin{array}{lll} +\sqrt{t'}, & +\sqrt{t''}, & -\sqrt{t'''} \\ +\sqrt{t'}, & -\sqrt{t''}, & +\sqrt{t'''} \\ -\sqrt{t'}, & +\sqrt{t''}, & +\sqrt{t'''} \\ -\sqrt{t'}, & -\sqrt{t''}, & -\sqrt{t'''} \end{array}$$

and these combinations give the following values for x', x'', x''', x^{IV}:

$$x'=\frac{A+\sqrt{t'}+\sqrt{t''}-\sqrt{t'''}}{4}$$

$$x''=\frac{A+\sqrt{t'}-\sqrt{t''}+\sqrt{t'''}}{4}$$

$$x'''=\frac{A-\sqrt{t'}+\sqrt{t''}+\sqrt{t'''}}{4}$$

$$x^{IV}=\frac{A-\sqrt{t'}-\sqrt{t''}-\sqrt{t'''}}{4}$$

Example. In the equation $x^4 - 3x^3 - 15x^2 + 19x + 30 = o$, $A=3, B=-15, C=-19$; $D=30$. These values, when substituted in the transformed equation, give

$$t^3 - 147t^2 + 3171t - (+55)^2$$

The roots of this equation are 1, 25, 121. Now since here $A^3 - 4AB + 8C = +55$, positive consequently, ∴ the four first values for x', x'', x''', x^{IV}, must be taken, and these give $x' = 5$, $x'' = -3$, $x''' = -1$, $x^{IV} = +2$.

SECTION XLIII.

Prob. From the given equation of the indeterminate nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = o,$$

find another equation for the squares of its roots.

Solution 1. Let the required equation have the roots $x'^2, x''^2, x'''^2, x^{IV2}$, &c. it must consequently be composed of the n simple equations $t - x'^2 = o$, $t - x''^2 = o$, $t - x'''^2 = o$, &c. Hence we have, as in § XXXIV, the transformed equation

$$t^n - (2)t^{n-1} + (2^2)t^{n-2} - (2^3)t^{n-3} + (2^4)t^{n-4}) + \ldots\ldots$$
$$\ldots\ldots \pm (2^{n-1})t \mp (2^n) = o$$

The numerical expressions may either be taken immediately from the Tables, or may be found by the method given in the two preceding chapters.

2. But this equation may also be found by the second method in § XXXIV. Put for instance, $x^2 = 1$, ∴ $x = \sqrt{t}$, and substitute this value of x in the given

equation. Here there are two distinct cases; viz. one when n is even, the other when n is odd.

3. First, let $n=2m$. Arrange the given equation thus:

$$x^{2m}+Bx^{2m-2}+Dx^{2m-4}+Fx^{2m-6}+\&c.$$
$$=Ax^{2m-1}+Cx^{2m-3}+Ex^{2m-5}+Gx^{2m-7}+\&c.$$

Substitute $\surd t$ for x; this gives

$$t^m+Bt^{m-1}+Dt^{m-2}+Ft^{m-3}+\&c.$$
$$=(At^{m-1}+Ct^{m-2}+Et^{m-3}+Gt^{m-4}+\&c.)\ \surd t;$$

or, when both sides of this equation are squared, and the terms properly arranged,

$$t^{2m}+(2B-A^2)\,t^{2m-1}+(2D-2AC+B^2)\,t^{2m-2}$$
$$+(2F-2AE+2BD-C^2)\,t^{2m-3}+\&c.=0.$$

4. Let $n=2m+1$. In this case, when $\surd t$ is put for x, we have

$$(t^m+Bt^{m-1}+Dt^{m-2}+Ft^{m-3}+\&c.)\ \surd t$$
$$=At^m+Ct^{m-1}+Et^{m-2}+\&c.$$

and when both sides of the equation are squared, and the terms properly arranged, we obtain the same equation as in 3, except that we get $2m+1$, instead of $2m$.

5. For both cases of the equation we consequently have

$$t^n+(2B-A^2)\,t^{n-1}+(2D-2AC+B^2)\,t^{n-1}$$
$$+(2F-2AE+2BD-C^2)\,t^{n-2}+\&c.=0.$$

Remark. If we had not already known how to find the expressions (2), (2^2), (2^3), &c. by another method, we could have found them immediately by these means, by placing the equations in 1 and 5, opposite one another, and

putting the coefficients of the same powers of t equal to one another. Thus for instance we obtain

$$
\begin{aligned}
-[2] &= 2B - A^2 \\
[2^2] &= 2D - 2AC + B^2 \\
-[2^3] &= 2F - 2AE + 2BD - C^2 \\
[2^4] &= 2H - 2AG + 2BF - 2CF + D^2 \\
&\&c.
\end{aligned}
$$

from which the law is easily seen.

If we denote the coefficients of the given equation, in order to denote the places which they occupy in it, by $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}, \overset{4}{A}$, &c. instead of A, B, C, D, &c. the law will be still more easily perceived. Then we have

$$
\begin{aligned}
-[2] &= 2\overset{2}{A} - \overset{1}{A}\overset{1}{A} \\
[2^2] &= 2\overset{4}{A} - 2\overset{3}{A}\overset{1}{A} + \overset{2}{A}\overset{2}{A} \\
-[2^3] &= 2\overset{6}{A} - 2\overset{5}{A}\overset{1}{A} + 2\overset{4}{A}\overset{2}{A} - \overset{3}{A}\overset{3}{A} \\
[2^4] &= 2\overset{8}{A} - 2\overset{7}{A}\overset{1}{A} + 2\overset{6}{A}\overset{2}{A} + \overset{5}{A}\overset{3}{A}\,\overset{4}{A}\overset{4}{A}
\end{aligned}
$$

and in general

$$
\begin{aligned}
\pm[2^n] &= 2\overset{2n}{A} - 2\overset{2n-1}{A}\overset{1}{A} + 2\overset{2n-2}{A}\overset{2}{A} - 2\overset{2n-3}{A}\overset{3}{A} + \ldots \\
&\ldots\ldots \mp \overset{n+1}{A}\overset{n-1}{A} \pm \overset{n}{A}\overset{n}{A}
\end{aligned}
$$

the upper sign obtains when n is even, and the lower when n is odd.

Euler uses these formulæ for finding the impossible roots of an equation [Complete Introduction to the Differential Calculus, translated by Michelsen, Part III. p. 135], but he gives no proof of them, but merely says, that they may be found by the theory of combinations. A proof of these formulæ different from the above, may be seen in Klügel's Mathematical Dictionary, Art. Combination, p. 469, &c.

SECTION XLIV.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0,$$

find the equation for the mth powers of its roots.

Solution. The roots of the required equation are x'^m, x''^m, x'''^m, &c. Hence we obtain in the same way as in the foregoing §, the equation

$$t^n - [m]\,t^{n-1} + [m^2]\,t^{n-2} - [m^3]\,t^{n-3} + [m^4]\,t^{n-4} - \ldots\ldots$$
$$\ldots\ldots \pm [m^{n-1}]\,t \mp [m^n] = 0$$

The numerical expressions here are all of the form $[a^\beta]$, and .·. may easily be found.

SECTION XLV

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0,$$

find the equation for the differences of its roots.

Solution 1. The number of the different values which the function $x' - x''$ contains by the substitution and transposition of the roots, is equal to the number of variations (in the sense in which Hindenburg uses this word)* of n different things of the second class, .·. $= n\,.\,(n-1)$. The required equation is consequently of the $n\,.\,(n-1)$th degree. Further it is evident from § XXXVI, that this equation contains only even powers of t, and that .·., when for the sake of brevity we put $n\,.\,(n-1) = 2m$, it has the following form:

$$t^{2m} - A't^{2m-2} + B't^{2m-4} - C't^{2m-6} + \&c. = 0;$$

* See vol. 1. p. 83, *note.*

and the roots of these equations are $(x'-x'')^2$, $(x'-x''')^2$, $(x'-x^{\mathrm{v}})^2$, &c., $(x''-x''')^2$, $(x''-x^{\mathrm{v}})^2$, &c. &c.

The coefficients A', B', C', &c. may be determined in the same way as in § XXXVI. yet the calculation by this method is attended with many difficulties, and besides, the law of the terms cannot be easily discovered. The following method, which will be frequently used in the sequel, is more simple and general.

2. For this purpose, put

$$S1 = (x'-x'')^2 + (x'-x''')^2 + (x'-x^{\mathrm{v}})^2 + \ldots + (x''-x''')^2 + \ldots$$
$$S2 = (x'-x'')^4 + (x'-x''')^4 + (x'-x^{\mathrm{v}})^4 + \ldots + (x''-x''')^4 + \ldots$$
$$S3 = (x'-x'')^6 + (x'-x''')^6 + (x'-x^{\mathrm{v}})^6 + \ldots + (x''-x''')^6 + \ldots$$
&c.

then the expressions $S1$, $S2$, $S3$, &c. are no other than the sum, the sum of the squares, the sum of the cubes &c., of the roots of the equation for t.

3. Since the expressions $S1$, $S2$, $S3$, &c. are the same for the transformed equation, as the expressions (1), (2), (3), &c. are for the given equation, consequently the formulæ found in § IX. are equally applicable to the coefficients A', B', C', &c., when throughout $S1$, $S2$, $S3$, &c. are put for (1), (2), (3), &c. and also $-A'$, $+B'$, $-C'$, for A, B, C, &c. Thus we have

$$A' = S1$$
$$B' = \frac{A'S1 - S2}{2}$$
$$C' = \frac{B'S1 - A'S2 + S3}{3}$$
$$D' = \frac{C'S1 - B'S2 + A'S3 - S4}{4}$$
&c.

If we had calculated the expressions $S1$, $S2$, $S3$, &c., we should have found, by means of these equations, the coefficients A', B', C', D', &c.

4. If the expressions $S1$, $S2$, $S3$, &c. are solved, we obtain

$$S1 = (n-1)(x'^2 + x''^2 + x'''^2 + \&c.) - 2(x'x'' + x'x''' + x''x''' + \&c.)$$
$$= (n-1)(2) - 2(1^2)$$
$$S2 = (n-1)(x'^4 + x''^4 + x'''^4 + \&c.) - 4(x'x''^3 + {}'^3x'' + x'x'''^3 + x'^3x''' + x''x'''^3 + x''^3x''' + \&c.) + 6(x'^2x''^2 + x'^2x'''^2 + x''^2x'''^2)$$
$$= (n-1)(4) - 4(13) + 6(2^2)$$
$$S3 = (n-1)(x'^6 + x''^6 + x'''^6 + \&c.) - 6(x'x''^5 + x'^5x'' + x'x'''^5 + x'^5x''' + x''x'''^5 + x''^5x''' + \&c.) + 15(x'^2x''^4 + {}'^4x''^2 + x'^2x'''^4 + x'^4x'''^2 + x''^2x'''^4 + x''^4x'''^2 + \&c.) - 20(x'^3x''^3 + x'^3x'''^3 + x''^3x'''^3 + \&c.)$$
$$= (n-1)(6) - 6(15) + 15(24) - 20(3^2)$$
&c.

These values of $S1$, $S2$, $S3$, &c. need only be substituted in the equations in 3, in order to find the coefficients A', B', C', &c.

5. But these values may at any time be reduced, by means of the two equations

$$(\alpha\beta) = (\alpha)(\beta) - (\alpha + \beta)$$
$$2(\alpha^2) = (\alpha)^2 - (\overline{2\alpha})$$

to sums of powers only, and then we have

$$S1 = (n-1)(2) - 2\left[\frac{(1)^2 - (2)}{2}\right]$$

$$S2 = (n-1)(4) - 4[(1)(3) - (4)] + 6\left[\frac{(2)^2 - (4)}{2}\right]$$

$$S3 = (n-1)(6) - 6[(1)(5) - (6)] + 15[(2)(4) - (6)] - 20\left[\frac{(3)^2 - (6)}{2}\right]$$

&c.

or after the proper reduction,

6.

$$S1 = n(2) - 2\frac{(1)^2}{2}$$

$$S2 = n(4) - 4(3)(1) + 6\frac{(2)^2}{2}$$

$$S3 = n(6) - 6(5)(1) + 15(4)(2) - 20\frac{(3)^2}{2}$$

and in general

$$S\mu = n(2\mu) - 2\mu(2\mu - 1(1) + \frac{2\mu.2\mu-1}{1.2}(2\mu-2)(2) - \ldots\ldots \ldots\ldots + \frac{2\mu.2\mu-1.2\mu-2\ldots\ldots\mu+1}{1.2.3\ldots\ldots\mu}\frac{(\mu)^2}{2}$$

These formulæ will be of great use in the sequel in the theory of imaginary roots of equations.

Remark. From Elementary Books on Algebra it is known, that when in the equation $x^n - Ax^{n-1} + \&c.$, $x + \frac{A}{n}$ is substituted for x, its second term vanishes. But since by this substitution, all the roots of this equation are diminished in an algebraical sense by the magnitude $\frac{A}{n}$, $\therefore$ their differences remain the same, consequently also the equation for their differences undergoes no change.

We obtain, however, this advantage by losing the second term, that the values of the expressions (1), (2), (3), &c. $S1$, $S2$, $S3$, &c. are much more simple. Thus we have

$$\begin{aligned}
(1) &= 0\\
(2) &= -2B\\
(3) &= 3C\\
(4) &= 2B^2 - 4D\\
(5) &= -5BC + 5E\\
(6) &= -2B^3 + 3C^2 + 6BD\\
&\&c.
\end{aligned}$$

Further, from 6, because $(1) = 0$, we get

$$\begin{aligned}
S1 &= n(2)\\
S2 &= n(4) + 3(2)^2\\
S3 &= n(6) + 15(4)(2) - 10(3)^2\\
&\&c.
\end{aligned}$$

These formulæ may be used with advantage in the case, when in the values of $S1$, $S2$, $S3$, &c., which have been found in 4, there are such numerical expressions as exceed the limits of the annexed tables. If this be not the case, it will be better to retain the former formulæ.

SECTION XLVI.

Prob. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation for the sums of every two of its roots.

Solution 1. The required equation must have the following roots:

$x' + x''$, $x' + x'''$, $x' + x^{IV}$, ... $x'' + x'''$, $x'' + x^{IV}$......

and the number of these roots is equal to the number of

combinations of n things taken two and two, consequently $=\frac{n \,.\, n-1}{1 \,.\, 2}$, for which, for the sake of brevity, I shall put m. The required equation will consequently be of the mth degree; it may be represented by

$$t^m - A't^{m-1} + B't^{m-2} - C't^{m-3} + \&c. = o.$$

The coefficients A', B', C', &c. may be most easily determined from the method used in the foregoing §.

2. For this purpose, then, put

$S1 =$
$(x'+x'') + (x'+x''') + (x'+x^{v}) + \ldots + (x''+x''') + \ldots$
$S2 =$
$(x'+x'')^2 + (x'+x''')^2 + (x'+x^{v})^2 + \ldots + (x''+x''')^2 + \ldots$
$S3 =$
$(x'+x'')^3 + (x'+x''')^3 + (x'+x^{v})^3 + \ldots + (x''+x''')^3 + \ldots$
&c.

so that the expressions $S1$, $S2$, $S3$, &c. denote the sums of the first, second, third, &c. powers of the roots. If we have determined the values of these equations in any way, then the coefficients in 3 of the foregoing § give the coefficients A', B', C', &c.

3. If we solve these expressions, we get

$$\begin{aligned}
S1 &= (n-1)\,(1) \\
S2 &= (n-1)\,(2) + 2(1^2) \\
S3 &= (n-1)\,(3) + 3(12) \\
S4 &= (n-1)\,(4) + 4(13) + 6(2^2) \\
S5 &= (n-1)\,(5) + 5(14) + 10(23)
\end{aligned}$$

&c.

whence the law may very easily be discovered.

4. If we wish to represent the values of the expressions $S1$, $S2$, $S3$, &c. immediately in sums of powers, we only need proceed as in 5 of the foregoing §. We then obtain

$$S1 = (n-1)\,(1)$$
$$S2 = (n-1)\,(2)+2\left[\frac{(1)^2-(2)}{2}\right]$$
$$S3 = (n-1)\,(3)+3\,[(1)\,(2)-(3)]$$
$$S4 = (n-1)\,(4)+4\,[(1)\,(3)-(4)]+6\left[\frac{(2)^2-(4)}{2}\right]$$
$$S5 = (n-1)\,(5)+5[(1)(4)-(5)]+10[(2)(3)-(5)]$$
&c.

or after the proper reduction

$$S1 = (n-1)\;(1)$$
$$S2 = (n-2)\;(2) + 2\frac{(1)^2}{2}$$
$$S3 = (n-2^2)\,(3) + 3\,(2)\,(1)$$
$$S4 = (n-2^3)\,(4) + 4\,(3)\,(1) + 6\frac{(2^2)}{2}$$
$$S5 = (n-2^4)\,(5) + 5\,(4)\,(1) + 10\,(3)\,(2)$$
&c.

and in general

$$S\mu=(n-2^{\mu-1})\,(\mu)+\mu\,(\mu-1)\,(1)+\frac{\mu\,.\,\mu-1}{1\,.\,2}(\mu-2)\,(2)$$
$$+\frac{\mu\,.\,\mu-1\,.\,\mu-2}{1\,.\,2\,.\,3}(\mu-3)\,(3)+\ldots\ldots$$

and the last term of this series is either

$$\frac{\mu\,.\,\mu-1\,.\,\mu-2\,\ldots\ldots\ldots\,\frac{\mu}{2}+1}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\,\frac{\mu}{2}}\;\frac{(\frac{\mu}{2})}{2}$$

or

$$\frac{\mu \cdot \mu-1 \cdot \mu-2 \ldots\ldots \frac{\mu+3}{2}}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots \frac{\mu-1}{2}} \left(\frac{\mu+1}{2}\right)\left(\frac{\mu-1}{2}\right)$$

according as m is an even or an odd number.

The abbreviations in the note to the preceding §, may, moreover, be also applied here.

SECTION XLVII.

Prob. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation for the function $(ax' + bx'')^p$; when p is a whole positive number.

Solution 1. Since in the function $(ax' + bx'')^p$, we can put every other root of the given equation for the roots x', x'', consequently the number of the values, which the function can contain, $= n \cdot n - 1$, for which I shall substitute m. The required equation is consequently only of the mth degree, and may $\therefore$ be represented by the equation

$$t^m - A't^{m-1} + B't^{m-2} - C't^{m-3} + \&c. = 0.$$

2. The method which has been made use of in the two preceding sections, for determining the coefficients A', B', C', &c. may likewise be applied here. Thus, if we denote by $S1$, $S2$, $S3$, &c. the sums of the first, second, third, &c. powers of the roots of the transformed equation, we have

$$S1 = (ax' + bx'')^{p} + (ax'' + bx')^{p} + (ax' + bx''')^{p} + \ldots$$
$$S2 = (ax' + bx'')^{2p} + (ax'' + bx')^{2p} + (ax' + bx''')^{2p} + \ldots$$
$$S3 = (ax' + bx'')^{3p} + (ax'' + bx')^{3p} + (ax' + bx''')^{3p} + \ldots$$
&c.

Having determined these equations, the equations in 3, § 45, give the values of the coefficients A', B', C'. It is now only necessary to determine the expression $S1$; having found this, we then obtain the remaining ones, $S2$, $S3$, &c. when we successively substitute $2p$, $3p$, &c. for p.

To find $S1$, we shall arrive most readily at the object in the following way.

3. Make the new expression ...... (ϕ) ......
$$\Sigma = (ax' + bz)^p + (ax'' + bz)^p + (ax''' + bz)^p + (ax^{IV} + bz)^p + \text{\&c.}$$
in which z denotes any unknown magnitude hitherto undetermined. If we solve this expression by means of the binomial theorem according to the powers of z, we obtain
$$\Sigma = a^p\,(x'^p + x''^p + x'''^p + x^{IV p} + \text{\&c.}) + pa^{p-1}b\,(x'^{p-1} + x''^{p-1} + x'''^{p-1} + x^{IV p-1}\ \text{\&c.})\,z + \text{\&c.}$$
or .... (ψ) ....
$$\Sigma = a^p(p) + pa^{p-1}b(p-1)\,.\,z + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2(p-2)\,.\,z^2 + \text{\&c.}$$
This equation must always be true, whatever we substitute for z.

4. Now, if we successively put x', x'', x''', x^{IV}, &c. for

z, and denote that which Σ becomes by these means, by Σ', Σ'', Σ''', Σ^{iv}, &c., we have in the first place from the equation (ϕ)

$$\Sigma' = (a+b)^p x'^p + (ax''+bx')^p + (ax'''+bx')^p + \&c.$$
$$\Sigma'' = (ax'+bx'')^p + (a+b)^p x''^p + (ax'''+bx'')^p + \&c.$$
$$\Sigma''' = (ax'+bx''')^p + (ax''+bx''')^p + (a+b)^p x'''^p + \&c.$$
$$\&c.$$

Hence it follows that

$$\Sigma' + \Sigma'' + \Sigma''' + \&c. = (a+b)^p[p] + S1$$

Further, from the equation (ψ) we obtain

$$\Sigma' = a^p[p] + pa^{p-1}b[p-1]\,.\,x' + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2[p-2]\,.\,x'^2 + \&c.$$

$$\Sigma'' = a^p[p] + pa^{p-1}b[p-1]\,.\,x'' + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2[p-2]\,.\,x''^2 + \&c.$$

$$\Sigma''' = a^p[p] + pa^{p-1}b[p-1]\,.\,x''' + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2[p-2]\,.\,x'''^2 + \&c.$$

$$\&c.$$

and hence

$$\Sigma' + \Sigma'' + \Sigma''' + \&c. = na^p[p] + pa^{p-1}b[p-1]\,[1] + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2[p-2]\,[2] + \&c.$$

5. If we put the two values found for $\Sigma' + \Sigma'' + \Sigma'''$&c. equal to one another, we obtain

$$[a+b]^p\,[p] + S1 = na^p[p] + pa^{p-1}b[p-1][1] + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2\,[p-2]\,[2] + \&c.$$

and hence

$$S1 = [na^p - (a+b)^p]\,[p] + pa^{p-1}b\,[p-1]\,[1] + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2\,[p-2]\,[2] + \frac{p\,.\,p-1\,.\,p-2}{1\,.\,2\,.\,3}a^{p-3}b^3[p-3]\,[3]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{p\,.\,p-1\,.\,p-2\ldots 1}{1\,.\,2\,.\,3\ldots\ldots p}b^p[0]\,[p]$$

6. If in this expression for $S1$, we combine the first and the last terms, the second and the last but one, and generally every two terms, of which one is as distant from the first, as the other is from the last, and at the same time keep in mind that $[0] = x'^0 + x''^0 + x'''^0 + \&c. = n$, we then obtain

$$\begin{aligned} S1 = {} & (n(a^p + b^p) - (a+b)^p)\,[p] + \\ & p\,(a^{p-1}b + ab^{p-1})\,[p-1]\,[1] + \\ & \frac{p\,.\,p-1}{1\,.\,2}(a^{p-2}b^2 + a^2b^{p-2})\,[p-2]\,[2] + \\ & \frac{p\,.\,p-1\,.\,p-2}{1\,.\,2\,.\,3}(a^{p-3}b^3 + a^3b^{p-3})\,[p-3]\,[3] + \\ & \&c. \end{aligned}$$

The last term of the expression, when p is an even number, is

$$\frac{p\,.\,p-1\ldots\ldots\frac{p}{2}+1}{1\,.\,2\ldots\ldots\ldots\ldots\frac{p}{2}}\,.\,a^{\frac{p}{2}}\,.\,b^{\frac{p}{2}}\,.\,\left[\tfrac{p}{2}\right]$$

But if p be an odd number, then the last term is

$$\frac{p\,.\,p-1\ldots\frac{p+3}{2}}{1\,.\,2\ldots\frac{p-1}{2}}\left(a^{\frac{p+1}{2}}\,.\,b^{\frac{p-1}{2}} + a^{\frac{p-1}{2}}\,.\,b^{\frac{p+1}{2}}\right)\cdot\left[\tfrac{p+1}{2}\right]\left[\tfrac{p-1}{2}\right]$$

7. If in the expression for $S1$, we substitute for p,

$2p$, $3p$, $4p$, &c. successively, we obtain the values of the expressions $S2$, $S3$, $S4$, &c. and the substitution of these values in the formulæ in 3, § 45, gives the values of the assumed coefficients $A,'$ B', C', &c.

SECTION XLVIII.

Prob. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = o$$

find the equation for the function $(ax' + bx'')^{-p}$, when p is a whole positive number.

Solution. From the equation found in the preceding § for t, which has the roots $(ax' + bx'')^p$, $(ax'' + bx')^p$, $(ax' + bx''')^p$, &c. another may be derived, which has (§ 10) the reciprocal roots $(ax' + bx'')^{-p}$, $(ax'' + bx')^{-p}$, $(ax' + bx''')^{-p}$, &c. and this will be the equation which is here sought.

SECTION XLIX.

From the foregoing problems it is sufficiently seen, what must be done, in order to find the equation on which a given function of the roots of an equation depends. By these, then, we arrive merely at the two following points:

1. To find all the possible values of which the given function is capable.

2. From these values to form the required equation. I shall begin with the first.

In order to find all the possible values of a function,

we must transpose its roots in as many different ways as possible with the other roots of the given equation, and with each other; and of all the results or values thus obtained, we only retain those, which are actually different from one another.

If all the roots of a given equation are in a function, it is only necessary to permute these roots in all possible ways. Consequently, if the given equation be of the nth degree, then a function of this kind generally contains $1 . 2 . 3 \ldots n$ values, because n things can be permuted this number of times. But if the form of the function is such, that more permutations than one generate equal results, then the number of the values is often less; and if all the values be equal, then the function is symmetrical.

If in the function there is only a number μ of the n roots of the given equation, then these n roots enter into the function in as many different ways as there are combinations in n things taken μ and μ; and the number of these combinations

$$= \frac{n . n - 1 . n - 2 \ldots\ldots n - \mu + 1}{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots \mu}$$

Every such combination, however, allows of $1 . 2 . 3 \ldots \mu$ permutations of the roots it contains; consequently the number of the values which such a function generally contains is

$$= n . n - 1 . n - 2 \ldots\ldots n - \mu + 1$$

$\therefore$ it is equal to the number of the variations of n things taken μ and μ. But if amongst these there are equal values, then this number will often be much less; although in

the assumed case they never can be less than n, because the number of variations never can be less than the number of elements. Consequently the transformed equation in this case can never be of a lower degree than the given one itself.

In the general inquiry respecting functions, it is always allowable to assume, that all the roots of the given equation are contained in them, because in the contrary case, only each of the roots which are wanting, considered with a coefficient $=0$, can be added to the function.

SECTION I.

EXPLANATION 1. Functions are said to be homogeneous, when they contain the same roots, and when in all the transpositions of these roots, they either at the same time change, or remain the same.

Let, for instance, the functions

$$x' + x'' - x''' - x^{iv},\ x'x'' - x'''x^{iv}$$

and likewise the functions

$$\frac{x''}{x'''} + \frac{x'''}{x'} + \frac{x'}{x''},\ x'^p x''^q x'''^r + x''^p x'''^q x'^r + x'''^p x'^q x''^r$$

be homogeneous. Then the first two have no more than the following 6 different and corresponding values:

$$\begin{array}{ll}
x' + x'' - x''' - x^{iv}, & x'x'' - x'''x^{iv} \\
x' + x''' - x'' - x^{iv}, & x'x''' - x''x^{iv} \\
x' + x^{iv} - x'' - x''', & x'x^{iv} - x''x''' \\
x''' + x^{iv} - x' - x'', & x'''x^{iv} - x'x'' \\
x'' + x^{iv} - x' - x''', & x''x^{iv} - x'x''' \\
x'' + x''' - x' - x^{iv}, & x''x''' - x'x^{iv}
\end{array}$$

and the two last no more than 2, viz.

$$\frac{x''}{x'''} + \frac{x'''}{x'} + \frac{x'}{x''},\ x'^{p}x''^{q}x'''^{r} + x''^{p}x'''^{q}x'^{r} + x'''^{p}x'^{q}x''^{r}$$

$$\frac{x'''}{x''} + \frac{x''}{x'} + \frac{x'}{x'''},\ x'^{p}x'''^{q}x''^{r} + x'''^{p}x''^{q}x'^{r} + x''^{p}x'^{q}x'''^{r}$$

2. The letter f prefixed to the roots of an equation, or even to other magnitudes, in the sequel, always denotes a rational function of these roots or magnitudes. Thus $f:\ (x')$ denotes a rational function of x', $f:\ (x')(x'')$ a rational function of x' and x'', and in general $f:\ (x')\ (x'')\ (x''')\ (x'^{v})\ \ldots\ldots (x^{(\mu)})$ a rational function of $x', x'', x''', x'^{v}, \ldots\ldots x^{(\mu)}$, and so in like manner of other magnitudes. In order to distinguish the functions, sometimes also the letters F, ϕ, ψ are made use of instead of f.

In this notation of the functions, it is preferable with each set of brackets ( ), which follows the letter f, to attach a definite representation of the manner in which the magnitudes contained in it are combined with the others; so that when any permutation of these magnitudes under the symbol f is intended, one of the permutations corresponding to it in the expression represented by it, must be denoted.

Thus, if $f:\ (x')\ (x'')\ (x''') = (x'x'' - x''')\ (x''-x')$, we then have

$$\begin{aligned}
f:\ (x')\ (x''')\ (x'') &= (x'x'''-x'')\ (x'''-x') \\
f:\ (x'')\ (x')\ (x''') &= (x''x'-x''')\ (x'-x'') \\
f:\ (x'')\ (x''')\ (x') &= (x''x'''-x')\ (x'''-x'') \\
f:\ (x''')\ (x')\ (x'') &= (x'''x'-x'')\ (x'-x''') \\
f:\ (x''')\ (x'')\ (x') &= (x'''x''-x')\ (x''-x''')
\end{aligned}$$

3. In order to distinguish the values which a given function contains by the permutation of its magnitudes, from the symbols by which this permutation is denoted, I shall call the latter *types*. Thus, the function $f:(x')(x'')(x''')$ has six types, viz.: $f:(x')(x'')(x''')$, $f:(x')(x''')(x'')$, $f:(x'')(x')(x''')$, $f:(x'')(x''')(x')$, $f:(x''')(x')(x'')$, $f:(x''')(x'')(x')$; and generally there are always as many types of a function as permutations of the magnitudes under the functional symbol.

The types are, as it were, the representatives of the values which a function contains, and in general investigations of functions may be used with very great advantage. If, for instance, we wished to show that any particular function had such a form, that the values arising from these or those transpositions were equal to one another, instead of actually expressing these permutations, which would often be attended with a great deal of trouble, it is only necessary to give the types, which correspond to the equal values.

SECTION LI.

When a function has such a form, that any two of its values are equal to one another, then the function must always necessarily have more than two equal values. Thus, if the function be such, that

$$f:(x')(x'')(x''')=f:(x')(x''')(x'')$$

then also must

$$f:(x'')(x')(x''')=f:(x'')(x''')(x')$$
$$f:(x''')(x')(x'')=f:(x''')(x'')(x')$$

For the first equation shows, that the value of the type $f:(x')\,(x'')\,(x''')$ remains unchanged, when the roots of the two last sets of brackets are transformed, ∴ the types $f:(x'')\,(x')\,(x''')$, $f:(x''')\,(x')\,(x'')$ in the same transformation of the brackets, must remain unchanged, because the equality, in the sense in which it is here taken, by which is meant no more than identity, is not derived from the numerical nature of the roots, but merely from the nature of their combination, consequently from the form of the functions.

SECTION LII.

Auxiliary Rule. If we combine a series of elements $a, b, c, d, e, \ldots\ldots p$, with as many numbers 1, 2, 3, 4, $5\ldots\ldots\pi$, arranged in any order according to a cipher, for instance, as follows:

$$\begin{matrix} 3 & 2 & 1 & 6 & 4 & 5 & 7 & 8 & 9 \ldots\ldots \pi \\ a & b & c & d & e & f & g & h & i \ldots\ldots p \end{matrix}$$

after this, permute the elements in the manner denoted by the ciphers placed over them, and from the permutation thus obtained, derive another, from this again another, and generally from every permutation last found, derive a new one, always observing in the transposition the law denoted by the ciphers: now I affirm, that by continual permutation, we must necessarily return again to the first permutation.

Thus we obtain from the permutation $A_1 = \overset{583214796}{abcdefghi}$ for the figures placed over it no more than 9 permutations $A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}$

$$\begin{array}{ll} & \overline{5\,8\,3\,2\,1\,4\,7\,9\,6} \\ A_1 & = a\,b\,c\,d\,e\,f\,g\,h\,i \\ A_2 & = e\,h\,c\,b\,a\,d\,g\,i\,f \\ A_3 & = a\,i\,c\,h\,e\,b\,g\,f\,d \\ A_4 & = e\,f\,c\,i\,a\,h\,g\,d\,b \\ A_5 & = a\,d\,c\,f\,e\,i\,g\,b\,h \\ A_6 & = e\,b\,c\,d\,a\,f\,g\,h\,i \\ A_7 & = a\,h\,c\,b\,e\,d\,g\,i\,f \\ A_8 & = e\,i\,c\,h\,a\,b\,g\,f\,d \\ A_9 & = a\,f\,c\,i\,e\,h\,g\,d\,b \\ A_{10} & = e\,d\,c\,f\,a\,i\,g\,b\,h \end{array}$$

If we proceed with the last permutation in the same manner as we did with the preceding ones, we again obtain the first.

Proof. Let

$A_1, A_2, A_3, A_4, \ldots\ldots\ldots A_\mu, \ldots\ldots\ldots A_\nu, \ldots\ldots\ldots$

denote the permutations which may successively be derived from the expression $A_1, = a\,b\,c\,d\,e\,f \ldots\ldots p$ according to the law of any one cipher.

Since the number of transpositions, which an expression can generally contain is always limited, we must $\therefore$ necessarily once come to a permutation A_ν, which is equal to one of the preceding A_μ. But if $A_\nu = A_\mu$, consequently also $A_{\nu-1} = A_{\mu-1}$; for if the permutations $A_{\nu-1}, A_{\mu-1}$, were not equal to one another, then also the permutations A_ν, A_μ, could not be equal, since A arises from $A_{\nu-1}$ by the same transformation of the elements as A_μ arises from $A_{\mu-1}$. In the san way we may further conclude from $A_{\nu-1} = A_{\mu-1}$, that also $A_{\nu-2} = A_{\mu-2}$, and hence again, that $A_{\nu-3} = A_{\mu-3}$, and so on. Consequently

$A_{\nu-(\mu-1)} = A_{\mu-(\mu-1)} = A_1$. We have $\therefore$ a permutation $A_{\nu-(\mu-1)}$ which is equal to the first. Q. E. D.

The contents of all the permutations obtained according to a given rule of transposition, are, for the sake of shortening the expression, called a period, because we always obtain the same permutations again, however long we continue the transposition.

SECTION LIII.

From the foregoing § we deduce the following propositions.

I.—That all the permutations of a period are different from one another.

For if in the period $A_1, A_2, A_3, A_4 \ldots\ldots A_\mu \ldots\ldots A_\nu \ldots\ldots A_\omega$ there are two equal permutations A_μ, A_ν, then also must $A_{\nu-1} = A_{\mu-1}$, $A_{\nu-2} = A_{\mu-2}$, and so on; consequently also $A_{\nu-(\mu-1)} = A_{\mu-(\mu-1)} = A_1$; but in this case A_ω could not be the last permutation of the period.

II.—Let B denote any permutation different from A_1, now it may belong to the period $A_1, A_2, A_3, \ldots\ldots A_\omega$ or not; further, let B_2, B_3, B_4, &c. be the permutations derived from B_1, according to the same rule of transposition by which A_2, A_3, A_4, &c. was derived from A_1; I affirm, that in this case the two periods arising from A_1 and B_1 consist of the same number of permutations.

For since the rule of transposition, which is denoted by the figures, does not refer to the elements themselves, but only to their places, so it is quite the same, in reference to the number of the permutations of which a

period consists, which elements are in the different places of the first permutation.

III.—If B_1 is equal to any one of the permutations $A_1, A_2, A_3 \dots\dots\dots A_\omega$ of the first period, then the two periods consist of the same permutations.

This proposition is an immediate consequence of the foregoing §.

IV.—If B_1 be not equal to any of the permutations $A_1, A_2, A_3, \dots\dots A_\omega$ of the first period, then the permutations of the two periods are all different from one another.

For if in the period $B_1, B_2, B_3 \dots\dots B_\varkappa \dots\dots B_\omega$ there be any permutation $B_\varkappa$, which is equal to a permutation A_μ of the period $A_1, A_2, A_3 \dots\dots A_\mu \dots\dots A_\omega$, then likewise must $B_{\varkappa+1} = A_{\mu+1}$, because $B_{\varkappa+1}$ is derived from $B_\varkappa$ by the same rule as $A_{\mu+1}$ is from A_μ; and when we further conclude in this way that $B_{\varkappa+2} = A_{\mu+2}$, $B_{\varkappa+3} = A_{\mu+3}$, and so on, lastly $B_{\omega+1} = A_{\mu+\omega-\varkappa+1}$. But since $B_{\omega+1} = B_1$, $A_{\mu+\omega-\varkappa+1} = A_{\mu-\varkappa+1}$, then must $B_1 = A_{\mu-\varkappa+1}$, which is contrary to the supposition that B_1 is different from all the permutations contained in the first period.

SECTION LIV.

Prob. Let a function be such, that any two given types are equal to one another: find all the equal values of the function which arise from this supposition (§ 51).

Solution. For the sake of perspicuity, I shall confine

myself to a single case, because it will be sufficiently clear from it how we are to proceed in every other one.

1. Let

$$f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$$

denote any function for which the two types

A_1 .... $f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$

A_2 .... $f:(x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'')$

are equal to one another. Compare these types, and observe how the roots are transposed in the first, second, third, fourth, and fifth brackets, when A_2 is deduced from A_1. If we retain this transposition in our memory, and then derive from A_2 a new type by the same rule by which A_2 was generated from A_1, from this last derive another, and continue this proceeding till we return again to the first, we shall obtain the following period, consisting of five types:

A_1 .... $f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$

A_2 .... $f:(x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'')$

A_3 .... $f:(x^{v})\,(x')\,(x'')\,(x''')\,(x^{\prime v})$

A_4 .... $f:(x'')\,(x''')\,(x^{\prime v})\,(x^{v})\,(x')$

A_5 .... $f:(x^{\prime v})\,(x^{v})\,(x')\,(x'')\,(x''')$

2. These types must necessarily be all equal, because they have all been derived from one another by the same rule. Now, since the function $f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$ has exactly as many types as there are transpositions in five magnitudes, consequently 120 types, it only remains that we take from the remaining 115 those which, under the hypothesis that $A_1 = A_2$, are equal to one another.

3. But this can be done very easily. For we only require to take away from the remaining types any one whatever, and from this derive a second period by the same rule, then again from the 110 types which still remain take away another, and by the same rule form a third period, and continue this till there are none remaining.

4. In this way we obtain 24 periods, each of which consists of five equal types. But if these be already found, then also the equal values, as soon as they are known, of the function itself corresponding to them, can be found.

Example. Suppose we have observed, that the function $x'x''^2x'''^3 + x'''x^{IV2}x^{V3} + x^{V}x'^2x''^3 + x''x'''^2x^{IV3} + x^{IV}x^{V2}x'^2$ remains the same, when in the terms in which the roots $x', x'', x''', x^{IV}, x^{V}$ are found, we put the roots $x''', x^{IV}, x^{V}, x', x''$ respectively, or more briefly $f: (x')(x'')(x''')(x^{IV})(x^{V}) = f: (x''')(x^{IV})(x^{V})(x')(x'')$. Since this is exactly the equation which was used in the solution for the illustration of the operation, it is $\therefore$ certain, that the function has 24 times five equal values. Thus the period in 1 contains the five following values:

$$f: (x')(x'')(x''')(x^{IV})(x^{V}) =$$
$$x'x''^2x'''^3 + x'''x^{IV2}x^{V3} + x^{V}x'^2x''^3 + x''x'''^2x^{IV3} + x^{IV}x^{V2}x'^3$$
$$f: (x''')(x^{IV})(x^{V})(x')(x'') =$$
$$x'''x^{IV2}x^{V3} + x^{V}x'^2x''^3 + x''x'''^2x^{IV3} + x^{IV}x^{V2}x'^3 + x'x''^2x'''^3$$
$$f: (x^{V})(x')(x'')(x''')(x^{IV}) =$$
$$x^{V}x'^2x''^3 + x''x'''^2x^{IV3} + x^{IV}x^{V2}x'^3 + x'x''^2x'''^3 + x'''x^{IV2}x^{V3}$$
$$f: (x'')(x''')(x^{IV})(x^{V})(x') =$$

$$x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime V3} + x^{\prime V}x^{V2}x^{\prime 3} + x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime 3} + x^{\prime\prime\prime}x^{\prime V2}x^{V3} + x^{V}x^{\prime 2}x^{\prime\prime 3}$$

$$f:\ (x^{\prime V})\,(x^{V})\,(x^{\prime})\,(x^{\prime\prime})\,(x^{\prime\prime\prime}) =$$

$$x^{\prime V}x^{V2}x^{\prime 3} + x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime 3} + x^{\prime\prime\prime}x^{\prime V2}x^{V3} + x^{V}x^{\prime 2}x^{\prime\prime 3} + x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime V3}$$

which are evidently all equal.

SECTION LV.

PROB. Let there be a function such, that more than two of its types are equal: find the equal values of this function.

Solution. Let A, B, C, D, &c. denote the types, which, according to the hypothesis, are equal. In order from hence to find the equal values, proceed as follows.

1. First permute the type A, or even any other, according to the rule of transposition, that $A=B$, as was shown in the foregoing §. If the period, which we obtain from this, consist of μ types, then we have at once μ equal types.

2. Then permute each of these μ types in particular by the rule $A=C$. I shall assume that there are μ' types; then we have generally $\mu\mu'$ types, which are all equal.

3. Permute again each of the $\mu\mu'$ types obtained by the rule $A=D$, which may give μ'' types; then we have in all $\mu\mu'\mu''$ equal types.

4. In the same way we proceed, when we successively make use of the rules $A=E$, $A=F$, &c. $B=C$, $B=D$, &c. $C=D$, $C=E$, &c. or, in short, when we put the given

types A, B, C, &c. taken two and two in all possible ways, equal to one another.

5. Let the number of types, obtained according to the directions in 1, 2, 3, 4, $=\nu$. Take now from all the types, which arise from all the possible transpositions of x', x'', x''', &c. any other, which is not amongst those already found, and proceed with this according to the same directions, then we obtain ν equal types. If we continue this proceeding, till all the types are exhausted, we at length obtain a number of divisions, each consisting of ν equal types. But that the types in every such division are different from the types in all the other divisions, is an immediate consequence of § 53. IV.

Corollary. It follows from this solution, that the number of the different values which a function can contain, is always a sub-multiple of the number of all the values, which arise from all the transpositions of x', x'', x''', &c.

Example I. Let there be a function such, that

$$f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x')\ (x^{\prime v}) \\ = f:\ (x'')\ (x''')\ (x^{\prime v})\ (x'):$$

required to find its equal values.

From the equation $A=B$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x')\ (x^{\prime v})$, we obtain, in the first place, the period

$$f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x')\ (x^{\prime v}) \\ = f:\ (x''')\ (x')\ (x'')\ (x^{\prime v})$$

From each of these three types we obtain, by the application of the equation $A=C$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x^{\prime v})\ (x')$, a period of four equal values, ∴ in all twelve equal types, viz.

$$\begin{aligned}
&f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x^{\prime v})\ (x') = \\
&f:\ (x''')\ (x^{\prime v})\ (x')\ (x'') = f:\ (x^{\prime v})\ (x')\ (x'')\ (x''') = \\
&f:\ (x''')\ (x')\ (x^{\prime v})\ (x'') = f:\ (x')\ (x^{\prime v})\ (x'')\ (x''') = \\
&f:\ (x^{\prime v})\ (x'')\ (x''')\ (x') = f:\ (x'')\ (x''')\ (x')\ (x^{\prime v}) = \\
&f:\ (x''')\ (x')\ (x'')\ (x^{\prime v}) = f:\ (x')\ (x'')\ (x^{\prime v})\ (x''') = \\
&f:\ (x'')\ (x^{\prime v})\ (x''')\ (x') = f:\ (x^{\prime v})\ (x''')\ (x')\ (x'')
\end{aligned}$$

At length we obtain from these twelve types, by means of the equation $B=C$, or $f:\ (x'')\ (x''')\ (x')\ (x^{\prime v}) = f:\ (x'')\ (x''')\ (x^{\prime v})\ (x')$, consequently by the permutation of the roots in the two last brackets, twelve other types, which, together with the former, give all the twenty-four types of the function $f:\ (x')\ (x'')\ (x''')\ (x^{\prime v})$. Hence it follows, that a function of the supposed nature must necessarily be symmetrical.

EXAMPLE. II. Let a function be such, that

$$\begin{aligned}
&f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x')\ (x''')\ (x^{\prime v}) = \\
&f:\ (x')\ (x'')\ (x^{\prime v})\ (x''') = f:\ (x''')\ (x^{\prime v})\ (x')\ (x''):
\end{aligned}$$

required to find its equal values.

The equation $A=B$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x')\ (x''')\ (x^{\prime v})$ gives no more than these two equal types. If to these we apply the equation $A=C$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime v}) = f:\ (x')\ (x'')\ (x^{\prime v})\ (x''')$, we obtain the four following equal types:

$$\begin{aligned}
&f:\ (x')\ (x'')\ (x''')\ ({}^{\prime v}) = f:\ (x')\ (x'')\ (x^{\prime v})\ (x''') = \\
&f:\ (x'')\ (x')\ (x''')\ (x^{\prime v}) = f:\ (x'')\ (x')\ (x^{\prime v})\ (x''') =
\end{aligned}$$

From these we again obtain, by means of the equation $A=D$, or $f:\ (x')\ (x'')\ (x''')\ (x^{iv}) = f:\ (x''')\ (x^{iv})\ (x')\ (x'')$ the following eight equal types:

$$f:\ (x')\ (x'')\ (x''')\ (x^{iv}) = f:\ (x''')\ (x^{iv})\ (x')\ (x'') =$$
$$f:\ (x')\ (x'')\ (x^{iv})\ (x''') = f:\ (x^{iv})\ (x''')\ (x')\ (x'') =$$
$$f:\ (x'')\ (x')\ (x''')\ (x^{iv}) = f:\ (x''')\ (x^{iv})\ (x'')\ (x') =$$
$$f:\ (x'')\ (x')\ (x^{iv})\ (x''') = f:\ (x^{iv})\ (x''')\ (x'')\ (x') =$$

The equations $B=C$, $B=D$, $C=D$ always again give the same types. The type $f:\ (x')\ (x'')\ (x''')\ (x^{iv})$ consequently contains no more than seven equal types.

If now we take any other type of the remaining sixteen, viz. $f:\ (x''')\ (x')\ (x^{iv})\ (x'')$ and proceed with it as we previously did with $f:\ (x')\ (x'')\ (x''')\ (x^{iv})$, we again obtain eight equal types, viz.

$$f:\ (x''')\ (x')\ (x^{iv})\ (x'') = f:\ (x^{iv})\ (x'')\ (x''')\ (x') =$$
$$f:\ (x''')\ (x')\ (x'')\ (x^{iv}) = f:\ (x'')\ (x^{iv})\ (x''')\ (x') =$$
$$f:\ (x')\ (x''')\ (x^{iv})\ (x'') = f:\ (x^{iv})\ (x'')\ (x')\ (x''') =$$
$$f:\ (x')\ (x''')\ (x'')\ (x^{iv}) = f:\ (x'')\ (x^{iv})\ (x')\ (x''') =$$

If from among the eight remaining types, we select any one, for instance, $f:\ (x'')\ (x''')\ (x^{iv})\ (x')$, and proceed with it in the same way, we obtain them all.

Hence .·. it follows, that the function of the supposed kind can have no more than three different values, viz.: $(x')\ (x'')\ (x''')\ (x^{iv})$, $f:\ (x''')\ (x')\ (x^{iv})\ (x'')$, $f:\ (x'')\ (x''')\ (x^{iv})\ (x')$, and that consequently such a function will not lead to any equation higher than the third degree. Of this nature are the functions $(x'+x''-x'''-x^{iv})^2$, $x'x''+x'''x^{iv}$, and innumerable others; the method to find which will be given hereafter. But if we wish to use a function of this kind to solve equations of the fourth

degree, it is not sufficient that the transformed equation should be of a lower degree than the given one; but we must likewise be enabled, from the known values of such a function, to determine the roots x', x'', x''', $x^{\prime v}$ by equations of a lower degree than the fourth, because otherwise the transformation of the equation will be of no use. In the functions $x'x''+x'''x^{\prime v}$, $(x'+x''-x'''-x^{\prime v})^2$, this is actually the case, as we have already seen in § 41 and 42. In the sequel the conditions will be given, under which it is generally possible, from the known value of a function $f: (x')(x'')(x''')\ldots\ldots(x^{(\mu)})$ to find the values of the roots x', x'', x''', ... $x^{(\mu)}$ by equations of a lower degree than the mth.

SECTION LVI.

Prob. Determine the degree of the equation on which a given function depends.

Solution 1. If in a function $f: (x')(x'')(x''')\ldots\ldots(x^{(\mu)})$ there are all the roots of the given equation, and if this function be such, that in each transposition of its roots it changes its value, then the transformed equation is necessarily of the degree $1 \,.\, 2 \,.\, 3 \ldots\ldots \mu$.

2. If the function be such that a number of types A, B, C, D, &c. are equal, and if the number of the equal types which can in general be derived from them, by the directions given in 1, 2, 3, 4, of the foregoing §, $=\nu$, then the number of the different values, of which such a function is capable, or the degree of the transformed equation,

$$=\frac{1\,.\,2\,.\,3\ldots\ldots\ldots\mu}{\nu}$$

3. If the function remain the same, when m roots change their places in all possible ways, then the degree of the transformed equation

$$=\frac{1\,.\,2\,.\,3\ldots\ldots\mu}{1\,.\,2\ldots\ldots m}=\mu\,.\,\mu-1\ldots\ldots m+1$$

4. If the function still remain the same, when m' other roots, and again m'' other roots, and so on, change their places, then the degree of the transformed equation

$$=\frac{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mu}{1\,.\,2\ldots m\times1\,.\,2\ldots m'\times1\,.\,2\ldots m''\times\&c.}$$

5. If the function be such, that each time its value remains unchanged, when m roots, and again m' other roots, &c. change their places in all possible ways, and if, besides, a number of types A, B, C, D, &c. are equal, then the degree of the transformed equation

$$=\frac{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mu}{1\,.\,2\ldots m\times1\,.\,2\ldots m'\times1\,.\,2\ldots m''\times\&c.\times\nu}$$

when ν retains the signification it does in 2.

6. If all the roots of the given equation be not in the function $f:\ (x')\ (x'')\ (x''')\ldots(x^{(\mu)})$, and if the equation be of the nth degree, then all the formulæ given in 1, 2, 3, 4, 5, must be multiplied by the factor.

$$\frac{n\,.\,n-1\,.\,n-2\ldots\ldots\ldots n-\mu+1}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots\mu}$$

The reason of all this is sufficiently evident from what goes before.

SECTION LVII.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation on which the function $f\colon (x') \, (x'') \, (x''') \ldots (x^{(\mu)})$ depends.

Solution 1. Seek first all the different values which this function contains, both by substituting for its roots the other roots of the given equation, and by their permutation. Let these different values be denoted by y', y'', y''', $y^{\prime v} \ldots\ldots y^{(\pi)}$.

2. Then form the equation

$$(t - y')\,(t - y'')\,(t - y''') \ldots\ldots (t - y^{(\pi)}) = 0$$

and actually multiply the factors in the first part. Then

$$t^{\pi} - A't^{\pi-1} + B't^{\pi-2} - C't^{\pi-3} + \&c. = 0$$

is the equation obtained from this operation, $\therefore$

$$A' = y' + y'' + y''' + y^{\prime v} + \&c.$$
$$B' = y'y'' + y'y''' + y'y^{\prime v} + y''y''' + \&c.$$
$$C' = y'y''y''' + y'y''y^{\prime v} + y'y'''y^{\prime v} + \&c.$$
$$\&c.$$

3. The functions A', B', C', &c. are symmetrical in relation to y', y'', y''', ... $y^{(\pi)}$, and consequently no transposition of these magnitudes can effect any change in the values of the former functions. But the magnitudes y', y'', y''', ... $y^{(\pi)}$ are themselves again functions of the roots x', x'', x''', &c. and such too as merely transform

into one another, when the roots are transformed and permuted in every possible way. Consequently no further change takes place by transforming and permuting the roots in the above expressions for A', B', C', &c. than that y', y'', y''', &c. change their places. Now, since this effects no change in the values of A', B', C', &c. consequently these values remain unchanged by the transformation and permutation of x', x'', x''', &c. Therefore the coefficients A', B', C', &c. are necessarily symmetrical functions of the roots x', x'', x''', &c.

4. In the two first sections, however, it was shown, that every symmetrical function of the roots of an equation, may always be expressed rationally by the coefficients of this equation. Therefore also the coefficients A', B', C', &c. may always be expressed rationally by A, B, C, &c.

5. Consequently an equation may always be found, on which depends a given function of the roots of another equation, and the coefficients of the former will always be rational functions of the coefficients of the latter.

IV.—On Elimination, together with its Application to the Reduction of Equations.

Section LVIII.

Prob. Let there be n equations of the first degree given, which contain as many unknown magnitudes: required to reduce their solution to the solution of $n-1$ equations of the first degree only, which contain but $n-1$ of these unknown magnitudes.

Solution 1. Let

$$\begin{array}{l} ax + by + cz + \ldots\ldots + kv + lw = A \\ a_1x + b_1y + c_1z + \ldots\ldots + k_1v + l_1w = A_1 \\ a_2x + b_2y + c_2z + \ldots\ldots + k_2v + l_2w = A_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n-1}x + b_{n-1}y + c_{n-1}z + \ldots\ldots + k_{n-1}v + l_{n-1}w = A_{n-1} \end{array}$$

be the n given equations; $x, y, z \ldots\ldots v, w$, the n unknown magnitudes; $a, b, c, \ldots\ldots k, l$; $a_1, b_1, c_1, \ldots k_1, l_1$; $a_2, b_2, c_2, \ldots k_2, l_2$, &c.; likewise A_1, A_2, A_3, &c. the given magnitudes.

2. Assume $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots \Pi_{n-1}$ as $n-1$ magnitudes hitherto unknown, and multiply the second equation by Π_1, the third by Π_2, the fourth by Π_3, and so on; lastly, the last by Π_{n-1}; then add all these products to the first equation; hence arises the equation

$$\begin{aligned}
A+A_1\Pi_1 + A_2\Pi_2 + \dots\dots + A_{n-1}\Pi_{n-1} &= \\
(a+ a_1\Pi_1 + a_2\Pi_2 + \dots\dots + a_{n-1}\Pi_{n-1})\, x& \\
+(b+ b_1\Pi_1 + b_2\Pi_2 + \dots\dots + b_{n-1}\Pi_{n-1})\, y& \\
+(c+ c_1\Pi_1 + c_2\Pi_2 + \dots\dots + c_{n-1}\Pi_{n-1})\, z& \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots& \\
+(l+ l_1\Pi_1 + l_2\Pi_2 + \dots\dots + l_{n-1}\Pi_{n-1})\, w&
\end{aligned}$$

3. Now we try to determine the factors $\Pi_1, \Pi_2, \Pi_3, \dots \Pi_{n-1}$, in such a way, that the coefficients of all the unknown magnitudes $x, y, z, \dots\dots v, w$, those excepted which we wish to find vanish. If, for instance, we wished to find x, we put

$$\begin{aligned}
b + b_1\Pi_1 + b_2\Pi_2 + \dots\dots + b_{n-1}\Pi_{n-1} &= 0 \\
c + c_1\Pi_1 + c_2\Pi_2 + \dots\dots + c_{n-1}\Pi_{n-1} &= 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots& \\
l + l_1\Pi_1 + l_2\Pi_2 + \dots\dots + l_{n-1}\Pi_{n-1} &= 0
\end{aligned}$$

4. Hence the equation in 2 is reduced to the following one:

$$\begin{aligned}
A + A_1\Pi_1 + A_2\Pi_2 + \dots\dots + A_{n-1}\Pi_{n-1} &= \\
(a + a_1\Pi_1 + a_2\Pi_2 + \dots\dots + a_{n-1}\Pi_{n-1})\, x&
\end{aligned}$$

and this gives

$$x = \frac{A + A_1\Pi_1 + A_2\Pi_2 + \dots\dots + A_{n-1}\Pi_{n-1}}{a + a_1\Pi_1 + a_2\Pi_2 + \dots\dots + a_{n-1}\Pi_{n-1}}$$

and the determination of the assumed magnitudes $\Pi_1, \Pi_2, \Pi_3, \dots\dots \Pi_{n-1}$, depends upon the solution of the equations in 3.

5. In order $\therefore$ to find the value of x, we must solve the $n-1$ equations in 3, from it determine the $n-1$ mag-

nitudes Π_1, Π_2, Π_3, ... Π_{n-1}, and then substitute their values in the expression found for x.

6. If both in 3 and 4 we substitute a for b, we shall find y, and in the same manner we find z, when in 3 and 4 we substitute a for c, and so on.

REMARK. The reduction of equations here given, may sometimes be used with advantage, as will be seen by an example given further on in this work. But if we intend merely to solve the given equations, we shall by these means attain our object but very slowly; in this case the method in the following § is preferable.

SECTION LIX.

PROB. Let the following n equations of the first degree be given:

$$\begin{array}{l} a_1x + b_1y + c_1z + \ldots\ldots + k_1v + l_1w = m_1 \\ a_2x + b_2y + c_2z + \ldots\ldots + k_2v + l_2w = m_2 \\ a_3x + b_3y + c_3z + \ldots\ldots + k_3v + l_3w = m_3 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_nx + b_ny + c_nz + \ldots\ldots + k_nv + l_nw = m_n \end{array}$$

in which there are n unknown magnitudes $x, y, z, \ldots\ldots v, w$: find the values of these magnitudes directly, and without any substitution or any other calculation.

Solution 1. If we merely had the two equations with two unknown magnitudes

$$\begin{array}{l} a_1x + b_1y = m_1 \\ a_2x + b_2y = m_2 \end{array}$$

we should have found them in the usual way

$$x = \frac{m_1 b_2 - m_2 b_1}{a_1 b_2 - a_2 b_1}\ y = \frac{a_1 m_2 - a_2 m_1}{a_1 b_2 - a_2 b_1}$$

2. If we had the three equations with the three unknown magnitudes

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= m_1 \\ a_2 x + b_2 y + c_2 z &= m_2 \\ a_3 x + b_3 y + c_3 z &= m_3 \end{aligned}$$

we then should find

$$x = \frac{m_1 b_2 c_3 - m_1 b_3 c_2 - m_2 b_1 c_3 - m_2 b_3 c_1 + m_3 b_1 c_2 - m_3 b_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}$$

$$y = \frac{a_1 m_2 c_3 - a_1 m_3 c_2 - a_2 m_1 c_3 + a_2 m_3 c_1 + a_3 m_1 c_2 - a_3 m_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}$$

$$z = \frac{a_1 b_2 m_3 - a_1 b_3 m_2 - a_2 b_1 m_3 + a_2 b_3 m_1 + a_3 b_1 m_2 - a_3 b_2 m_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}$$

3. From the formulæ in 1 and 2, the rules for the solution of the above general equations follow by induction. In order to abbreviate them, I shall call the numbers which are affixed to the letters m, a, b, &c. symbolical numbers.

(*a*) Take the product $a_1 b_2 c_3 \ldots\ldots k_{n-1} l_n$; then permute the symbolical numbers in all possible ways, while the letters themselves are not changed; the aggregate of all these $1 \cdot 2 \cdot 3 \ldots\ldots n$ products, then gives the common denominator in the values of $x, y, z \ldots\ldots v, w$.

(*b*) In order to find the signs of every one of the terms,

of which the denominator consists, try how often in such a term a lower symbolical number follows a higher one, mediately or immediately. If the number of these successions be even or O, then the sign of the term is +; if it be odd, the sign is—.

(c) If the common denominator be found, we obtain from it the numerator in the value of x, merely by substituting m for a; the numerator in the value of y, by substituting m for b; the numerator in the value of z, by substituting m for c; and so in like manner with the other unknown magnitudes.

Thus the denominator in the values of x, y, z, is merely the product $a_1 b_2 c_3$, with the 1 . 2 . 3 permutations of the symbolical numbers; and with respect to the signs, if, for instance, the term $a_2 b_3 c_1$, has the sign +, because it contains two successions of a lower symbolical number to a higher, viz. 21, 31; but in the term $a_3 b_2 c_1$, there are three such successions, viz. 32, 31, 21, and this term consequently has the sign —. Likewise the numerators are formed in the manner given in (c).

Example. From the four equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1u &= m_1 \\ a_2x + b_2y + c_2z + d_2u &= m_2 \\ a_3x + b_3y + c_3z + d_3u &= m_3 \\ a_4x + b_4y + c_4z + d_4u &= m_4 \end{aligned}$$

we obtain for the common denominator in the values of x, y, z, u, the following expression:

$$\begin{aligned} &a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_3c_4d_2 \\ &+ a_1b_4c_2d_3 - a_1b_4c_3d_2 - a_2b_1c_3d_4 + a_2b_1c_4d_3 \\ &+ a_2b_3c_1d_4 - a_2b_3c_4d_1 - a_2b_4c_1d_3 + a_2b_4c_3d_1 \\ &+ a_3b_1c_2d_4 - a_3b_1c_4d_2 - a_3b_2c_1d_4 + a_3b_2c_4d_1 \\ &+ a_3b_4c_1d_2 - a_3b_4c_2d_1 - a_4b_1c_2d_3 + a_4b_1c_3d_2 \\ &+ a_4b_2c_1d_3 - a_4b_2c_3d_1 - a_4b_3c_1d_2 + a_4b_3c_2d_1 \end{aligned}$$

SECTION LX.

Since the values of the unknown magnitudes in the solution of the foregoing § always appear in the form of fractions, it may sometimes happen, that the common denominator $= 0$, as, for instance, in the two equations $a_1b_2 - a_2b_1 = 0$, and the three equations $a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 = 0$. In this case, if also the numerator $= 0$, then we arrive at expressions of the form $\frac{0}{0}$. Such a form as this merely indicates, that the conditions given in the equations are not such, that the values of the unknown magnitudes can be determined by their means alone. Thus, if we had the two equations $3x + 5y = 16$, $6x + 10y = 32$, then would $a_1 = 3, b_1 = 5, a_2 = 6, b_2 = 10, m_1 = 16, m_2 = 32$, and consequently from the formulæ in 1 of the foregoing §,

$$x = \frac{16 \,.\, 10 - 32 \,.\, 5}{3 \,.\, 10 - 6 \,.\, 5} = \frac{0}{0}, y = \frac{3 \,.\, 32 - 6 \,.\, 16}{3 \,.\, 10 - 6 \,.\, 5} = \frac{0}{0},$$

the values of x and y ∴ would remain undetermined. But we immediately see why they must continue to be undetermined. For if we divide the second equation by 2, we obtain the first; consequently the latter is contained in the former, and we have ∴ in fact no more than one equation, from which neither x nor y can be determined.

But if the given equations be such, that the denominator vanishes in the value of an unknown magnitude, but not the numerator, consequently that we arrive at an expression of the form $\frac{a}{0} = \infty$; then this result always indicates, that the relations expressed by the equation are contradictory, and cannot obtain at the same time, while the unknown magnitudes, as is always assumed here, have only finite values. Thus, suppose we have the two equations $3x + 5y = 16$, $6x + 10y = 20$, we obtain from 1 of the foregoing §, $x = \frac{60}{0}$, $y = \frac{-36}{0}$; consequently, as we are convinced that there can be no other values but these, there must be contradictory relations in the given equations. This, indeed, is really the case; for if we multiply the first equation by 2, we get $6x + 10y = 32$, whereas, from the second equation $6x + 10y = 20$.

SECTION LXI.

The problems, § LVIII. and LXIX. contain all that relates to the elimination of equations of the first degree. I shall now direct my attention to the elimination of equations of higher degrees; and I shall first assume, that there are no more than two equations given with two, or even more unknown magnitudes. Here two cases must be considered: first, when the first equation, in reference to the magnitudes to be eliminated, is of the first degree, and the second of a higher; secondly, when both equations are of higher degrees.

There is no difficulty in the first case; for we only

require to find from the first equation the value of the magnitude to be eliminated, and substitute this value in the second, when we obtain an equation, in which this magnitude does not occur.

In the second case, by multiplying by proper factors, and by the requisite combination of the results thus obtained, we always try to reduce the degree of the equations in reference to the magnitude to be eliminated, till we come to an equation, which contains only the first power of this magnitude. If from this equation we find the value of the magnitude to be eliminated, and substitute it in that equation in which it occurs in the lowest power, we shall obtain the required final equation.

The following problems will sufficiently elucidate the foregoing.

SECTION LXII.

Prob. Let p, q, r, p', q', r', be functions of y; further, let the two equations

$$\text{I. } p + qx + rx^2 = 0$$
$$\text{II. } p' + q'x + r'x^2 = 0$$

be given between x and y: find the equation, which arises from the elimination of x.

Solution 1. Multiply the first equation by p', the second by p, then subtract the results thus obtained from one another, and divide by x; hence arises the equation

$$pq' - p'q + (pr' - p'r)\,x = 0$$

and this gives

$$x = \frac{p'q - pq'}{pr' - p'r}.$$

2. Further, multiply the first equation by r', and the second by r, and subtract; then we have

$$pr' - p'r + (qr' - q'r)\, x = 0$$

If in this equation we substitute for x its value obtained from 1, we get the equation

$$(\psi) \ldots (pr'-p'r)^2 + (p'q-pq')\,(qr'-q'r) = 0$$

which only contains y, and which consequently is the final equation sought.

3. If we had immediately substituted the value of x from 1, in one of the given equations, for instance in I, we should have found

$$p + \frac{q\,(p'q-pq')}{pr' - p'r} + \frac{r\,(p'q-pq')^2}{(pr'-p'r)^2} = 0$$

and if we multiply by $(pr'-p'r)^2$, and then divide by p, we get the same equation as in 2.

SECTION LXIII.

PROB. From the two equations

$$\text{I. } p + qx + rx^2 = 0$$
$$\text{II. } p' + q'x + r'x^2 + s'x^3 = 0$$

eliminate x, supposing that p, q, r, p', q', r', s', are such expressions as do not contain x.

Solution 1. Multiply the first equation by p', the second by p, and subtract the results, and we get the equation

$$pq' - p'q + (pr' - p'r)\,x + ps'x^2 = 0$$

2. If we combine the equation I. with this one, the

case in the preceding § enters here; only that $pq'-p'q$, $pr'-p'r$, ps', are here what p', q', r', were in the former. We only require ∴ to substitute the former values for the latter in the equation (ψ) of the foregoing §. If this be actually done, we obtain the equation

$$(p^2s'+qrp'-prq')^2+(pqs'-prr'+r^2p')\times$$
$$(pqq'-q^2p'-p^2r'+prp') = o$$

3. If we solve this equation, and then divide by p, we obtain

$$p^3s'^2 + p^2rr'^2 + pr^2q'^2 + r^3p'^2 - qr^2p'q' +$$
$$(q^2-2pr)\ (rp'r'+pq's') + (3pqr-q^3)p's' -$$
$$pqrq'r' - p^2qr's' = o$$

an equation which does not contain x.

SECTION LXIV.

PROB. Again, let p, q, r, s, p', q', r', s', be functions which do not contain x: find the result of the elimination of x from the two equations

$$\text{I. } p + qx + rx^2 + sx^3 = o$$
$$\text{II. } p' + q'x + r'x^2 + s'x^3 = o$$

Solution 1. Multiply the first equation by p', the second by p, and subtract the results; after dividing by x, this gives

$$pq' - qp' + (pr' - rp')\,x + (ps' - sp')\,x^2 = o$$

2. Further, multiply the first equation by s', the second by s, and again subtract; this gives

$$sp' - ps' + (sq' - qs')\,x + (sr' - rs')\,x^2 = o.$$

3. It is not necessary to continue the reduction further; for since the equations found in 1 and 2 are similar to the equations I and II, in § 62, for which the result of the elimination was there found, it is only necessary in the equation (ψ) of that section, to make the following substitutions:

$$\begin{aligned} &pq'-qp' \text{ for } p, \quad sp'-ps' \text{ for } p' \\ &pr'-rp' \text{ for } q, \quad sq'-qs' \text{ for } q' \\ &ps'-sp' \text{ for } r, \quad sr'-rs' \text{ for } r' \end{aligned}$$

4. By this substitution we obtain, after the proper solution,

$$\begin{aligned} &(pq'-qp')^2\ (sr'-rs')^2 - 2(pq'-qp')\ (ps'-sp') \\ &(sp'-ps')\ (sr'-rs') + (ps'-sp')^2\ (sp'-ps')^2 \\ &+ (pr'-rp')^2\ (sp'-ps')\ (sr'-rs') - (pq'-qp') \\ &(pr'-rp')\ (sq'-qs')\ (sr'-rs') - (pr'-rp') \\ &(ps'-sp')\ (sp'-ps')\ (sq'-qs') + (pq'-qp') \\ &(ps'-sp')\ (sq'-qs')^2 = 0 \end{aligned}$$

5. The first part of this equation consists of seven terms, of which five are divisible by $sp'-ps'$. The other two, viz. the first and fifth, taken together, give

$$\begin{aligned} &(pq'-qp')\ (sr'-rs') \times \\ &[(pq'-qp')\ (sr'-rs') - (pr'-rp')\ (sq'-qs')] \\ &= (pq'-qp')\ (sr'-rs')\ (pqr's'+rsp'q'-prq's'-qsp'r') \\ &= (pq'-qp')\ (sr'-rs')\ (sp'-ps')\ (rq'-qr') \end{aligned}$$

and consequently the sum of these two terms is also divisible by $sp'-ps'$.

6. If $\therefore$ the equation in 4 be divided by $sp'-ps'$, we at last obtain the equation

$(pq' - qp')(sr' - rs')(rq' - qr') + 2(pq' - qp')$
$(sp' - ps')(sr' - rs') + (sp' - ps')^3 + (pr' - rp')^2$
$(sr' - rs') + (pr' - rp')(sp' - ps')(sq' - qs') -$
$(pq' - qp')(sq' - qs')^2 = 0$

SECTION LXV.

Prob. From the two equations

$$\text{I.}\quad p + qx + rx^2 + sx^3 + tx^4 = 0$$
$$\text{II.}\quad p' + q'x + r'x^2 + s'x^3 + t'x^4 = 0$$

eliminate the magnitude x.

Solution 1. Multiply the first equation by p', the second by p, and subtract; after dividing by x, this gives

$$pq' - qp' + (pr' - rp')x + (ps' - sp')x^2 + (pt' - tp')x^3 = 0$$

2. Further, multiply the first equation by t', the second by t, and again subtract; this gives

$$pt' - tp' + (qt' - tq')x + (rt' - tr')x^2 + (st' - ts')x^3 = 0$$

3. Since the equations in 1 and 2 are both of the third degree, in order to save the trouble of carrying on the operation, we need only use immediately the equation found in 6 of the foregoing §, when we make the following substitutions in it:

$pq' - qp'$ for p, $pt' - tp'$ for p'
$pr' - rp'$ for q, $qt' - tq'$ for q'
$ps' - sp'$ for r, $rt' - tr'$ for r'
$pt' - tp'$ for s, $st' - ts'$ for s'

and the result of this substitution is the final equation sought.

SECTION LXVI.

PROB. From the two general equations

I. $p + qx + rx^2 + sx^3 + \ldots\ldots + vx^m = 0$

II. $p' + q'x + r'x^2 + s'x^3 + \ldots\ldots + v'x^n = 0$

eliminate x.

Solution 1. I shall assume, that $m < n$. Multiply then one equation by the first term of the second, *i. e.* by p', and the other by the first term of the first, *i. e.* by p; then subtract, and divide the remainder by x; we then obtain an equation of the form

III. $A + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots + A_{-1} x^{n-1} = 0$

2. If again $m < n-1$, we proceed with the equations I and III, exactly as we did before with the equations I and II.

3. In this way we continue to diminish the degree of the resulting equation, till we arrive at an equation of the mth degree. Let

IV. $B + B_1 x + B_2 x^2 + B_3 x^3 + \ldots\ldots + B\ \ x^m = 0$

be this equation.

4. Combine the equation IV with the equation I in a two-fold way, viz. 1. when we multiply the first by p, the second by B; subtract the results, and divide the remainder by x; 2. when we multiply the first by v, the second by B_m, and again subtract the results from one

another. By this operation we obtain two equations of the $m-1$th degree.

$$\text{V.}\quad C+C_1x+C_2x^2+C_3x^3+\ \ldots\ldots\ +C_{m-1}x^{m-1}=0$$
$$\text{VI.}\quad D+D_1x+D_2x^2+D_3x^3+\ \ldots\ldots\ +D_{m-1}x^{m-1}=0$$

5. If we proceed with the two equations so obtained, in the same way as we did before with I and IV, we shall again obtain two equations of the $m-2$th degree. In this manner we reduce the degree of the equations less and less, till we arrive at two equations of the first degree. Let

$$K+K_1x=0\,,\ L+L_1x=0$$

be these two equations; then we have

$$x=-\frac{K}{K_1}=-\frac{L}{L_1}$$

and consequently

$$KL_1-LK_1=0$$

and this, since it does not contain x, is the final equation sought.

6. But it is by no means necessary to continue throughout the elimination to equations of the first degree; thus, if we have already found the results of the elimination for equations of a certain degree, then it is sufficient, as we have shown in the foregoing §, to continue the reduction to this degree only.

SECTION LXVII.

The method of elimination which has been applied in the foregoing §§, which Euler makes use of in the 19th

chapter of the 2nd Book of his Introduction, is perfectly general, but is essentially deficient in this respect, that it does not always give the final equation in its simple form. Thus, for instance, the equation in 2, § LXIII, was divisible by p, and then gave the equation in 3; in the same way, after dividing the equation in 4, § LXIV, by $sp'-ps'$ we obtained the equation in 6. This also obtains in the higher equations, and the divisors in this case are difficult to find. We shall however see, in the sequel, that these divisors are actually superfluous, and by no means belong to the final equation. Should we not ∴ probably be able to find such a divisor, we should have to solve not only a higher equation for y, than was actually required, but there would be also amongst its roots such as do not belong to the equations I and II of the foregoing §, consequently which were not so constituted, that we were enabled to find the corresponding values of x, which at the same time verify the two equations just mentioned.

Since the elimination of x from two equations between x and y has only this aim, to give one or more such values for y, that it may be possible to find one or more corresponding values for x, which are common to both equations, consequently every method which serves to attain this object, may also be applied to the elimination.

If ∴ we denote one of the values of x by a, which is common to both equations, then $x-a$ must be their common divisor; consequently it is merely necessary to find the conditions on which the possibility of a divisor of this kind depends. To effect this, we only require to proceed with the given equations, exactly as though we

sought their common divisor; the last remainder, which we get by the successive divisions, and which does not contain x, if it contains such a divisor, must necessarily vanish. If we put this remainder $= 0$, we then obtain the required conditional or final equation. The following problem will be sufficient to elucidate what has been said. The method to find the common divisor, is assumed to be known.

SECTION LXVIII.

Prob. From the two equations

$$\text{I.}\quad x^3 + 3x^2y + 3xy^2 - 98 = 0$$

$$\text{II.}\quad x^2 + 4xy - 2y^2 - 10 = 0$$

eliminate x by the method of the common divisor.

Solution. The calculation is as follows.

1. Dividend $x^3 + 3x^2y + 3xy^2 - 98$
   Divisor $x^2 + 4xy - 2y^2 - 10$
   Quotient $x - y$
   First Remainder $(9y^2 + 10)\,x - 2y^3 - 10y - 98$

2. Dividend $x^2 + 4xy - 2y^2 - 10$
   or rather $(9y^2 + 10)\,x^2 + 36xy^3 + 40xy -$
   $18y^4 - 110y^2 - 100$
   Divisor $(9y^2 + 10)\,x - 2y^3 - 10y - 98$
   Quotient $x + \dfrac{38y^3 + 50y + 98}{9y^2 + 10}$
   Second and last Remainder

$$-18y^4 - 110y^2 - 100 + \frac{(2y^3 + 10y + 98)(38y^3 + 50y + 98)}{9y^2 + 10}$$

3. If this remainder be put $=0$, and then multiplied by $9y^2+10$, we obtain

$$-86y^6-690y^4+3920y^3-1500y^2+5880y+8604=0$$

or, when we divide by two, and change all the signs

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0$$

and this is the conditional or final equation sought.

REMARK. Having found a value of y from the conditional equation, we can find the value of x, which corresponds to it, by substituting that value in the two equations I and II, and then seeking the common divisor. Thus we shall find, that $y=3$ verifies the final equation; if $\therefore$ we substitute this value in I and II, we obtain the two equations $x^3+9x^2+27x-98=0$, $x^2+12x-28=0$, whose common divisor is $x-2$. Accordingly $x=2$ is the value of x, which belongs to $y=3$.

But we could also have found this value of x directly; for we know already, that always when amongst the remainders obtained in the divisions for finding the common divisor, that remainder which $=0$, is considered as the last, the preceding remainder is the required divisor. If we apply this to the present case, then $(9y^2+10)x-2y^3-10y-98$ is this divisor; and if we put $y=3$, then this divisor is $91x-182$, or $x-2$, which agrees with the foregoing.

But hence it follows, that we must also have obtained the final equation, if we had substituted the value $x=\dfrac{2y^3+10y+98}{9y^2+10}$, which we obtained from the equation $(9y^2+10)x-2y^3-10y-98=0$, immediately in the

equation II, as the lowest of the two given ones. But if we make the same substitution in the equation I, we arrive at an equation of the ninth degree, which consequently contains one factor more of the third degree. But we shall see in the sequel, that this factor is actually superfluous, and that ∴ the equation in 3 is the complete final equation.

Since the final equation is of the sixth degree, there are ∴, besides the value $y=3$, five other values of y, for each of which there is a corresponding value of x. Consequently it verifies the equations I and II in six ways. The same equation of the sixth degree we could also have obtained from the equation in 3, § LXIII, if, as the present case requires, we had put $p=-2y^2-10$, $q=4y$, $r=1$, $p'=-98$, $q'=3y^2$, $r'=3y$ and $s'=1$.

SECTION LXIX.

From the method of the common divisor, we may derive another, which Euler gives in the work above mentioned.

Let

$$\text{I. } x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c. = 0$$
$$\text{II. } x^n + p'x^{n-1} + q'x^{n-2} + r'x^{n-3} + \&c. = 0$$

be the two equations, from which x is to be eliminated. If it happen that these two equations have any common divisor $x-a$, we can then put

$$\text{III. } x^m + px^{m-1} + qx^{m-2} + \&c. = (x-a)\,\Pi$$
$$\text{IV. } x^n + p'x^{n-1} + q'x^{n-2} + \&c. = (x-a)\,\Pi'$$

and then Π, Π' are the quotients, which arise from the

division of the first parts of the equations I, II by $x-a$. It is not necessary here actually to know these quotients; it is merely sufficient to observe, that they must necessarily have the following form:

$$\Pi = x^{m-1} + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \&c.$$
$$\Pi' = x^{n-1} + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \&c.$$

and that the first contains $m-1$ undetermined magnitudes A, B, C, &c., and the second $n-1$ undetermined magnitudes A', B', C', &c.

Now, if we eliminate $x-a$ from the two equations III, IV, we then obtain the identical equation

$$(\phi)\quad \begin{array}{l}(x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c.)\,\Pi' = \\ (x^n + p'x^{n-1} + q'x^{n-2} + r'x^{n-3} + \&c.)\,\Pi\end{array}$$

If we actually perform this multiplication, after having substituted for Π, Π', their assumed values, and then put the coefficients of the same powers of x in the two parts of the resulting equation equal to one another, we then obtain $m+n-1$ equations between the magnitudes p, q, r, &c. p', q', r', &c. A, B, C, &c. A', B', C', &c. which, in reference to the unknown magnitudes A, B, C, &c. A', B', C', &c. are all of the first degree only. Since $\therefore$ we have $m+n-1$ equations, and only $m+n-2$ undetermined magnitudes A, B, C, &c. A', B', C', &c. these can always be eliminated, and by this elimination we shall obtain an equation which contains no other magnitudes but the known functions p, q, r, &c. p', q', r', &c., and which consequently will be the required conditional or final equation.

The following problems will elucidate what has been said.

SECTION LXX.

Prob. From the two given equations

$$\text{I. } x^2 + px + q = 0$$

$$\text{II. } x^3 + p'x^2 + q'x + r' = 0$$

eliminate x by the method in the foregoing §.

Solution. Since here $m = 2$, and $n = 3$, then

$$\Pi = x + A,\ \Pi' = x^2 + A'x + B'$$

The equation (ϕ) of the foregoing § will ∴ be

$$(x^2 + px + q)(x^2 + A'x + B') =$$
$$(x^3 + p'x^2 + q'x + r')(x + A)$$

Hence, by actual multiplication and equating the coefficients, we obtain

$$A' + p = A + p'$$
$$B' + pA' + q = p'A + q'$$
$$pB' + qA' = q'A + r'$$
$$qB' = r'A$$

Since there are four equations here, and only three undetermined magnitudes A, A', B', we can ∴ eliminate these magnitudes, which only occur in the first power, and we shall then, after the proper reduction, obtain the same equation as in § LXIII, by substituting 1, p, q, 1, p', q', r', for r, q, p, s', r', q', p', respectively, as the form of the equations here given, compared with the others, requires.

SECTION LXXI.

Prob. From the given equations

$$\text{I. } x^3 + px^2 + qx + r = 0$$

$$\text{II. } x^4 + p'x^3 + q'x^2 + r'x + s' = 0$$

eliminate x by the method in § LXIX.

Solution. Here $m=3$, $n=4$; we have $\therefore$

$$\Pi=x^2+Ax+B,\ \Pi'=x^3+A'x^2+B'x+C'$$

If these values be substituted in the equation (ϕ) in § LXIX, we obtain

$$(x^3+px^2+qx+r)(x^3+A'x^2+B'x+C')=$$
$$(x^4+p'x^3+q'x^2+r'x+s')(x^2+Ax+B)$$

If we solve this equation, and equate the coefficients of its two parts, we get

$$\begin{aligned}
&A' + p = A + p'\\
&B' + pA' + q = B + p'A + q'\\
&C' + pB' + qA' + r' = p'B + q'A + r'\\
&pC' + qB' + rA' = q'B + r'A + s'\\
&qC' + rB' = r'B + s'A\\
&rC' = s'B
\end{aligned}$$

Since in these six equations there are only five undetermined magnitudes A, B, A', B', C', we can eliminate these, and thus we shall obtain an equation, which contains none but the unknown magnitudes p, q, r, p', q', r', s'', which consequently is the final equation sought.

SECTION LXXII.

Euler's second method, which is here elucidated, is at least quite as diffuse, if not more so, than the first; it is also quite as free as the first, from the fault of having superfluous factors, which we can easily convince ourselves of by the actual calculation of a few easy cases. Bezout, in his "Théorie générale des Equations Algébriques, Paris, 1799," has made use of a similar method; he has applied it to more than two equations, and to the elimination of more unknown magnitudes; he has shown

also, how we are to begin, in many cases, in order to find the complete final equation, without its including any thing superfluous. The work is, indeed, rather prolix, yet contains a great deal of matter, and is composed with much care. A revision of this work, with the application of the combination-analysis, would be a most useful undertaking.

From the hypothesis of the common divisor, we may also derive another method, which is not only much more simple, but also more direct, and more suited to the nature of equations than the preceding, inasmuch as it is founded on the theory of symmetrical functions. It has likewise this important advantage over the other one, that it always gives, at least for two equations and two unknown magnitudes, the complete final equation, without any thing heterogeneous.

For this purpose, we assume again the equations I, II, §LXIX. We suppose the equation I, with respect to x, is already solved; then its roots x', x'', x''', &c. are functions of its coefficients, and consequently also functions of y. In like manner we suppose the second solved; then its roots, which I shall denote by x', x'', x''', &c. are also functions of y. If the two equations have a common divisor, then must at least one of the roots of the first equation be equal to one of the roots of the second. Put $x'=\mathrm{x}'$, then $x'-\mathrm{x}'=0$ is the equation for y, which must be the case when the two determined roots x', x', are equal to one another. But since, equally well, every other two roots of both equations may be put equal to one another, we then obtain as many distinct equations $x'-\mathrm{x}'=0$, $x'-\mathrm{x}''=0$, $x'-\mathrm{x}'''=0 \ldots x''-\mathrm{x}'=0$, $x''-\mathrm{x}''=0$, $x''-\mathrm{x}'''=0 \ldots$

$x''' - \mathrm{x}' = 0$, $x''' - \mathrm{x}'' = 0$, $x''' - \mathrm{x}''' = 0$, &c. as the roots x', x'', x''', &c. may be combined with the roots x', x'' x''', &c. All these distinct equations, however, must at the same time be contained in the conditional or final equation sought, because there is no reason why it should exactly contain the one and not likewise the other; it must consequently be a product of these. The final equation is ∴ no other than

$$(\psi)\ldots\ldots \left\{\begin{matrix} (x' - \mathrm{x}')\ (x' - \mathrm{x}'')\ (x' - \mathrm{x}''') \ldots \times \\ (x'' - \mathrm{x}')\ (x'' - \mathrm{x}'')\ (x'' - \mathrm{x}''') \ldots \times \\ (x''' - \mathrm{x}')\ (x''' - \mathrm{x}'')\ (x''' - \mathrm{x}''') \ldots \times \\ \&c. \end{matrix}\right\} = 0$$

The first part of this equation undergoes no change, however we transform the roots x', x'', x''', &c.; it is ∴ symmetrical in reference to x', x'', x''', &c. But since it also suffers no change when we transform x', x'' x''', &c. it is also symmetrical with reference to x', x'', x''', &c. Consequently the first part of the equation (ψ) may always be expressed rationally by the coefficients of the given equations.

I shall hereafter show how we can give the equation (ψ) a more convenient form for calculation; but I shall first explain what has been stated, by a few problems.

SECTION LXXIII.

Prob. From the two equations

$$\text{I. } x^2 - Ax + B = 0$$
$$\text{II. } x^2 - A'x + B = 0$$

in which A, B, A', B', are given functions of y, eliminate x' by the method in the foregoing §.

Solution 1. The equation (ψ) of the foregoing §, since both equations are of the second degree, in this case is

$$(x'-\mathrm{x}')\,(x'-\mathrm{x}'')\,(x''-\mathrm{x}')\,(x''-\mathrm{x}'') = 0$$

or

$$[x'^2-\mathrm{x}'(\mathrm{x}'+\mathrm{x}'')+\mathrm{x}'\mathrm{x}'']\,[x''^2-x''(\mathrm{x}'+\mathrm{x}'')+\mathrm{x}'\mathrm{x}''] = 0$$

or, since $\mathrm{x}'+\mathrm{x}''=A'$, $x'x''=B'$

$$(x'^2 - x'A' + B')\,(x''^2 - x''A' + B') = 0$$

2. If we actually perform the multiplication, we obtain

$$x'^2x''^2 - (x'x''^2 + x'^2x'')\,A' + (x'^2 + x''^2)\,B'$$
$$+ x'x''A'^2 - (x' + x'')\,A'B' + B'^2 = 0$$

or since $x' + x'' = (1) = A$, $x'x'' = (1^2) = B$, $x'x''^2+x'^2x''=(12)=AB$, $x'^2+x''^2=(2)=A^2-2B$,

$$B^2-ABA'+(A^2-2B)\,B'+BA'^2-AA'B'+B'^2=0$$

and this equation is the final equation sought, which we should also have obtained, if we had eliminated in the usual way.

SECTION LXXIV.

Prob. From the given equations

$$\text{I. } x^3 - Ax^2 + Bx - C = 0$$
$$\text{II. } x^2 - A'x + B' = 0$$

eliminate x by the method in § LXXII.

Solution 1. The equation (ψ) § LXXII in this case, is

$$(x' - \mathrm{x}')\,(x' - \mathrm{x}'')\,(x'' - \mathrm{x}')\,(x'' - \mathrm{x}'')$$
$$(x''' - \mathrm{x}')\,(x''' - \mathrm{x}'') = 0$$

or

$$[x'^2-x'(\mathrm{x}'+\mathrm{x}'')+\mathrm{x}'\mathrm{x}'']\,[x''^2 - x''(\mathrm{x}'+\mathrm{x}'') + \mathrm{x}'\mathrm{x}'']$$
$$[x'''^2-x'''(\mathrm{x}' + \mathrm{x}'') + \mathrm{x}'\mathrm{x}''] = 0$$

or, since $x' + x'' = A'$, $x'x'' = B'$

$$(x'^2 - x'A' + B')\,(x''^2 - x''A' - B')$$
$$(x'''^2 - x'''A' + B') = 0$$

2. The actual multiplication of the three factors in the first part of this equation, gives

$$(1^3)^2 - (12^2)A' + (2^2)B' + (1^2 2)A'^2 - (12)A'B' +$$
$$(2)B'^2 - (1^3)A'^3 + (1^2)A'^2B' - (1)A'B'^2 + B'^3$$
$$= 0$$

or, when for the numerical expressions their values are substituted from the annexed tables,

$$C^2 - BCA' + (B^2 - 2AC)B' + ACA'^2 - (AB - 3C)A'B'$$
$$+ (A^2 - 2B)B'^2 - CA'^3 + BA'^2B' - AA'B'^2 + B'^3$$
$$= 0$$

which is the final equation sought.

SECTION LXXV.

Prob. From the given equations

$$\text{I. } x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \&c. = 0$$
$$\text{II. } x^n - A'x^{n-1} + B'x^{n-2} - C'x^{n-3} + \&c. = 0$$

eliminate x, by the method in § LXXII.

Solution 1. Since x', x'', x''', &c. are the roots of the equation II, then

$$(x - x')\,(x - x'')\,(x - x''')\ldots\ldots\ldots =$$
$$x^n - A'x^{n-1} + B'x^{n-2} - C'x^{n-3} + \&c.$$

If in this for x we substitute x', x'', x''', &c. successively, then

$$(x' - x')\,(x' - x'')\,(x' - x''')\ \&c. = x'^n - A'x'^{n-1} + \&c.$$
$$(x'' - x')(x'' - x'')(x'' - x''')\&c. = x''^n - A'x''^{n-1} + \&c.$$
$$(x''' - x')(x''' - x'')(x''' - x''')\&c. = x'''^n - A'x'''^{n-1} + \&c.$$

2. If these values be substituted in the equation (ψ), § 72, the latter is transformed into

$$\left.\begin{cases} \quad (x'^n - A'x'^{n-1} + B'x'^{n-2} - C'x'^{n-3} + \&c.) \\ \times\ (x''^n - A'x''^{n-1} + B'x''^{n-2} - C'x''^{n-3} + \&c.) \\ \times\ (x'''^n - A'x'''^{n-1} + B'x'''^{n-2} - C'x'''^{n-3} + \&c.) \\ \times\ \&c. \end{cases}\right\} =$$

The first part of this equation is no other than the product of all the expressions, which arise from substituting in the equation II for x all the roots x', x'', x''', &c. successively of the equation I.

3. But we immediately see, that the above product undergoes no change by any transformation of the roots x', x'', x''', &c. as in a transformation of this kind one factor is merely changed into another. The first part of the equation is ∴ necessarily a symmetrical function of the above roots, which consequently may always be omitted according to known rules. In this way we obtain an equation, which no longer contains x, and which ∴ is the final equation sought. From the operation itself, it follows besides, that it is complete, and contains nothing extraneous.

Remark. The problem from two given equations with two unknown magnitudes to eliminate one of these magnitudes, is consequently now solved in its most general form. The actual calculation involves many difficulties, and amongst these chiefly are the solution of the product in the first part of the equation in 2, and its reduction to numerical expressions. How these difficulties may be

removed by means of combinations, will be shown in the following §.

SECTION LXXVI.

PROB. Represent directly the result of the elimination of x from the two equations I and II of the foregoing §, fully solved.

Solution 1. To the equation II of the foregoing §, we can always, by dividing by its last term, give the following form:

$$1+(1)x+(2)x^2+(3)x^3+\ldots\ldots+(n)\,x^n=0$$

in which the coefficients (1), (2), (3), ... (n) denote given functions of y. This notation was chosen merely in order to facilitate the application of the combination-operations, and to make the law of the terms more evident. In order further to show, that two or more such coefficients are to be multiplied together, I shall merely put the numbers representing this operation in brackets near each other, and in these make use of repeating exponents. Thus ∴ (123), (2456), $(1^3 2^2)$, the first of these denotes the product of the coefficients (1), (2), (3); the second, the product of the coefficients (2), (4), (5), (6), and the third, the product of the third power of the coefficient (1) and the second power of the coefficient (2).

2. Put, as was done in the foregoing §, x', x'', x''', &c. successively for x, consequently the first part of the equation in 2 of that §, has the following form:

$$[1+(1)x' + (2)x'^2 + (3)x'^3 + (4)x'^4 + (5)x'^5 + \&c.]$$
$$\times[1+(1)x'' + (2)x''^2 + (3)x''^3 + (4)x''^4 + (5)x''^5 + \&c.]$$
$$\times[1+(1)x''' + (2)x'''^2 + (3)x'''^3 + (4)x'''^4 + (5)x'''^5 + \&c.]$$
$$\&c.$$

The number of factors here is equal to the degree of the equation I, $\therefore = m$.

3. First take the product of the two first factors, we then obtain

$$\begin{aligned} 1 &+ (1)(x'+x'') + (2) + (x'^2+x''^2) + (3)(x'^3+x''^3) \\ &\qquad + (1^2)(x'x'') \qquad + (12)(x'x''^2+x'^2x'') \\ &+ (4)(x'^4+x''^4) \qquad + (5)(x'^5+x''^5) \qquad + \&c. \\ &+ (13)(x'x''^3+x'^3x'') \qquad + (14)(x'x''^4+x'^4x'') \\ &+ (2^2)(x'^2x''^2) \qquad + (23)(x'^2x''^3+x'^3x''^2) \end{aligned}$$

4. Hence, if the equation I were of the second degree only, then this product must be represented by numerical expressions, as follow:

$$\begin{aligned} 1 &+ (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] + \&c. \\ &\qquad + (1^2)[1^2] + (12)[12] + (13)[13](+(14)[14] \\ &\qquad\qquad + (2^2)[2^2] + (23)[23] \end{aligned}$$

5. Now if we multiply the product in 3 by the third factor in 2, we then obtain, when the terms are properly arranged, the product

$$\begin{aligned} 1 &+ (1)(x'+x''+x''') + (2)(x'^2+x''^2+x'''^2) \\ &\qquad + (1^2)(x'x'' + x'x''' + x''x''') \\ &+ (3)(x'^3+x''^3+x'''^3) \\ &+ (12)\begin{pmatrix} x'x''^2 + x'^2x'' + x'x'''^2 + \\ x'^2x''' + x''x'''^2 + x''^2x''' \end{pmatrix} \\ &+ (1^3)(x'x''x''') \end{aligned}$$

$$+ (4)\ (x'^4 + x''^4 + x'''^4)$$
$$+ (13) \begin{pmatrix} x'x''^3 + x'^3x'' + x'x'''^3 + \\ x'^3x''' + x''x'''^3 + x''^3x''' \end{pmatrix}$$
$$+ (2^2)\ (x'^2x''^2 + x'^2x'''^2 + x''^2x'''^2)$$
$$+ (1^22)\ (x'x''x'''^2 + x'x''^2x''' + x'^2x''x''')$$

$$+ (5)\ (x'^5 + x''^5 + x'''^5)$$
$$+ (14) \begin{pmatrix} x'x''^4 + x'^4x'' + x'x'''^4 + \\ x'^4x''' + x''x'''^4 + x''^4x''' \end{pmatrix}$$
$$+ (23) \begin{pmatrix} x'^2x''^3 + x'^3x''^2 + x'^2x'''^3 + \\ x'^3x'''^2 + x''^2x'''^3 + x''^3x'''^2 \end{pmatrix}$$
$$+ (1^23)(x'x''x'''^3 + x'x''^3x''' + x'^3x''x''')$$
$$+ (12^2)\ (x'x''^2x'''^2 + x'^2x''x'''^2 + x'^2x''^2x''')$$
&c.

6. If $\therefore$ the equation I be only of the third degree, then the product in 2 may be represented as follows:

$$\begin{aligned} 1 + (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] + \&c. \\ + (1^2)[1^2] + (12)[12] + (13)[13] + (14)[14] \\ + (1^3)[1^3] + (2^2)[2^2] + (23)[23] \\ + (1^22)[1^22] + (1^23)[1^23] \\ + (12^2)[12^2] \end{aligned}$$

7. It is not necessary to continue the multiplication further, as we may very easily perceive the law from the products already found. Thus we see, immediately, that the figures in the parentheses and brackets are always the same, and compounded in the same way. The included numerical expressions merely denote this, that they are all the possible numerical divisors for

the numbers 1, 2, 3, 4, ... mn. I say all the possible numerical divisions; because some vanish, as, for example, in the products in 4 and 6; this merely proceeds from this, that the numerical expressions for such divisions in the assumed degree of the equation I, are not possible, because more roots are required for their formation than this equation can possibly have. Generally, when we have to do with particular equations, all those divisions vanish, for which either the numerical expressions, or the products of the coefficients do not obtain.

8. How all the possible divisions of the numbers may be easily found, without the chance of omitting one, may be seen by the combination-analysis. In order ∴ to find the elimination of x from the two equations,

I. $x^m - Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \&c. = 0$

II. $1 + (1)x + (2)x^2 + (3)x^3 + \ldots + (n)x^n = 0$

we must observe the following rules:

(a) Analyse the numbers 1, 2, 3, 4, ... mn in all possible ways.

(b) From every such analysis $\alpha\beta\gamma\delta \ldots$ make a term of the form $(\alpha\beta\gamma\delta\ldots)\ [\alpha\beta\gamma\delta\ldots]$.

(c) Then, if the sum of all these terms is represented by S, consequently

$$1 + S = 0$$

is the final equation sought.

9. All the numerical expressions relate to the equation I, and may partly be taken from the annexed tables, and partly may be calculated by the methods given in the two first sections. Nothing now remains but to elucidate this operation by a few examples.

SECTION LXXVII.

EXAMPLE 1. Find the result of the elimination of x from the two equations

I. $x^2 - Ax + B = 0$

II. $\mathfrak{A}x^5 + \mathfrak{B}x^4 + \mathfrak{C}x^3 + \mathfrak{D}x^2 + \mathfrak{E}x + \mathfrak{F} = 0.$

First give the equation II the form $1 + (1)x + (2)x^2 + (3)x^3 + (4)x^4 + (5)x^5$; consequently put $(1) = \frac{\mathfrak{E}}{\mathfrak{F}}$, $(2) = \frac{\mathfrak{D}}{\mathfrak{F}}$, $(3) = \frac{\mathfrak{C}}{\mathfrak{F}}$, $(4) = \frac{\mathfrak{B}}{\mathfrak{F}}$, $(5) = \frac{\mathfrak{A}}{\mathfrak{F}}$. Since the equation I is here only of the second degree, the divisions need not be continued further than the second class, because the numerical expressions for higher classes do not obtain (§ LXXVI, VII). The final equation ∴ has the following form:

$$\begin{aligned}0 = 1 &+ (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] \\ &+ (1^2)[1^2] + (12)[12] + (13)[13] + (14)[14] \\ &+ (2^2)[2^2] + (23)[23] \\ &+ (15)[15] + (25)[25] + (35)[35] + (45)[45] + (5^2)[5^2] \\ &+ (24)[24] + (34)[34] + (4^2)[4^2] \\ &+ (3^2)[3^2]\end{aligned}$$

Or, when we take the numerical expressions from the tables, for the numbers (1), (2), (3), (4), (5), substitute their values, and then multiply the whole equation by $\mathfrak{F}^2$,

$$
\begin{aligned}
0 = \mathfrak{F}^2 &+ \mathfrak{E}\mathfrak{F}A + \mathfrak{D}\mathfrak{F}(A^2-2B) + \mathfrak{C}\mathfrak{F}(A^3-3AB) \\
&\qquad + \mathfrak{E}^2B \qquad\qquad\quad + \mathfrak{D}\mathfrak{E}AB \\
&+\mathfrak{B}\mathfrak{F}(A^4-4A^2B+2B^2) + \mathfrak{A}\mathfrak{F}(A^5-5A^3B+5AB^2) \\
&+\mathfrak{C}\mathfrak{E}(A^2B-2B^2) \qquad\quad + \mathfrak{B}\mathfrak{E}(A^3B-3AB^2) \\
&+\mathfrak{D}^2B^2 \qquad\qquad\qquad\quad + \mathfrak{C}\mathfrak{D}AB^2 \\
&+\mathfrak{A}\mathfrak{E}(A^4B-4A^2B^2+2B^3) + \mathfrak{A}\mathfrak{D}(A^3B^2-3AB^3) \\
&+\mathfrak{B}\mathfrak{D}(A^2B^2-2B^3) \qquad\quad + \mathfrak{B}\mathfrak{C}AB^3 \\
&+\mathfrak{C}^2B^3 \\
&+\mathfrak{A}\mathfrak{C}(A^2B^2-2B^4)+\mathfrak{A}\mathfrak{B}AB^4+\mathfrak{A}^2B^5 \\
&+\mathfrak{B}^2B^4
\end{aligned}
$$

EXAMPLE II. Let the two equations

$$
\begin{aligned}
&\text{I.} \quad x^3 - Ax^2 + Bx - C = 0 \\
&\text{II.} \quad \mathfrak{A}x^3 + \mathfrak{B}x^2 + \mathfrak{C}x + \mathfrak{D} = 0
\end{aligned}
$$

be given.

If we reduce the equation II to the form $1 + (1)x + (2)x^2 + (3)x^3$; then $(1) = \frac{\mathfrak{C}}{\mathfrak{D}}$, $(2) = \frac{\mathfrak{B}}{\mathfrak{D}}$, $(3) = \frac{\mathfrak{A}}{\mathfrak{D}}$

Since in this case the equation I is of the third degree, we do not require to continue the divisions beyond the third class. The final equation ∴ has the form

$$
\begin{aligned}
0 = 1 &+ (1)[1] + (2)[2] + (3)[3] + (13)[13] + (23)[23] \\
&\qquad + (1^2)[1^2] + (12)[12] + (2^2)[2^2] + (1^23)[1^23] \\
&\qquad\qquad + (1^3)[1^3] + (1^22)[1^22] + (12^2)[12^2] \\
&+ (3^2)\,[3^2] + (13^2)[13^2] + (23^2)[23^2] + (3^3)[3^3] \\
&+ (123)[123] + (2^23)[2^23] \\
&+ (2^3)\,[2^3]
\end{aligned}
$$

or, when we substitute for the numerical expressions and for the numbers (1), (2), (3), their values, and then multiply by $\mathfrak{D}^3$

$$\begin{array}{lll}
0=\mathfrak{D}^3+\mathfrak{C}\mathfrak{D}^2A & +\mathfrak{B}\mathfrak{D}^2(A^2-2B) & +\mathfrak{A}\mathfrak{D}^2(A^3-3AB+3C) \\
 & +\mathfrak{C}^2\mathfrak{D}B & +\mathfrak{B}\mathfrak{C}\mathfrak{D}\,(AB-3C) \\
 & & +\mathfrak{C}^3C
\end{array}$$

$$\begin{array}{ll}
+\mathfrak{A}\mathfrak{C}\mathfrak{D}(A^2B-2B^2-AC) & +\mathfrak{A}\mathfrak{B}\mathfrak{D}(AB^2-2A^2C-BC) \\
+\mathfrak{B}^2\mathfrak{D}\,(B^2-2AC) & +\mathfrak{A}\mathfrak{C}^2\,(A^2C-2BC) \\
+\mathfrak{B}\mathfrak{C}^2AC & +\mathfrak{B}^2\mathfrak{C}BC
\end{array}$$

$$\begin{array}{ll}
+\mathfrak{A}^2\mathfrak{D}(B^3-3ABC+3C^2) & +\mathfrak{A}^2\mathfrak{C}(B^2C-2AC^2) \\
+\mathfrak{A}\mathfrak{B}\mathfrak{C}(ABC-3C^2) & +\mathfrak{A}\mathfrak{B}^2AC^2 \\
+\mathfrak{B}^3C^2 &
\end{array}$$

$$+\mathfrak{A}^2\mathfrak{B}BC^2+\mathfrak{A}^3C^3$$

In order to show the use of this formula in a particular case, I shall assume that the two equations

$$x^3-2ax^2+4ayx-y^3=0$$
$$ax^2+y^2x-ay^2=0$$

are given. If we equate these with the equations I, II, we find $A=2a$, $B=4ay$, $C=y^3$, $\mathfrak{A}=0$, $\mathfrak{B}=a$, $\mathfrak{C}=y^2$, $\mathfrak{D}=-ay^2$. Since here $\mathfrak{A}=0$, the foregoing equation is reduced to

$$\begin{array}{lll}
0=\mathfrak{D}^3+\mathfrak{C}\mathfrak{D}^2A & +\mathfrak{B}\mathfrak{D}^2(A^2-2B) & +\mathfrak{B}\mathfrak{C}\mathfrak{D}(AB-3C) \\
 & +\mathfrak{C}^2\mathfrak{D}B & +\mathfrak{C}^3C
\end{array}$$
$$\begin{array}{l}
+\mathfrak{B}^2\mathfrak{D}\,(B^2-2AC)\;+\;\mathfrak{B}^2\mathfrak{C}BC\;+\;\mathfrak{B}^3C^2 \\
+\mathfrak{B}^2\mathfrak{C}AC
\end{array}$$

and this equation obtains for every case, in which one of the given equations is of the third, and the other of the second degree. If in these we make the requisite substitutions, we obtain the required final equation

$$y^9+a^2y^7+6a^3y^6-12a^4y^5-12a^5y^4=0$$

or

$$y^5+a^2y^3+6a^3y^2-12a^4y-12a^5=0.$$

SECTION LXXVIII.

The method to find the final equation by means of the symmetrical functions, originated with Euler, who first made use of it in the Memoirs of the Berlin Academy for the year 1748, and applied it to a few easy examples. A short time after this, Cramer, in the second Appendix to his "Introduction à l'Analyse des Lignes Courbes," p. 660, &c. 1750, by a suitable method of notation (which I have partly adopted), made it more general, and the operation more easy. Both these great men chiefly endeavoured to prove by it, that two lines, one of which is of the mth, the other of the nth order, can be cut in no more than mn points. The proof of this does not belong to this work, but to the higher branches of geometry. It is quite enough to know, that this solely depends on the following rule, and is an easy and immediate consequence of it.

SECTION LXXIX.

Rule. When in the two equations

$$\text{I.}\quad x^m + \overset{1}{A}x^{m-1} + \overset{2}{A}x^{m-2} + \overset{3}{A}x^{m-3} + \ldots\ldots + \overset{m}{A} = 0$$

$$\text{II.}\quad x^n + \overset{1}{A}{}'x^{n-1} + \overset{2}{A}{}'x^{n-2} + \overset{3}{A}{}'x^{n-3} + \ldots\ldots + \overset{n}{A} = 0$$

between two magnitudes x, y, the coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, ... $\overset{m}{A}$, $\overset{1}{A}{}'$, $\overset{2}{A}{}'$, $\overset{3}{A}{}'$ ... $\overset{n}{A}{}'$, are merely whole rational functions of y, and that $\overset{1}{A}$ and $\overset{1}{A}{}'$ are of the first degree, $\overset{2}{A}$ and $\overset{2}{A}{}'$ of the second, $\overset{3}{A}$ and $\overset{3}{A}{}'$ of the third, and so on; then the final equation in y, which arises from the elimination of x, can never exceed the degree mn.

Proof 1. Let x', x'', x''', &c. be the roots of the equation I, then is $-[1]=\overset{1}{A}$, $[2]=\overset{1}{A}{}^2-2\overset{2}{A}$, $-[3]=\overset{1}{A}{}^3-3\overset{1}{A}\overset{2}{A}+3\overset{3}{A}$, and so on; from which we see, that [1] contains no other power of y than y^1, [2] no higher one than y^2, [3] no higher one than y^3; and it may also be satisfactorily proved with little trouble, from the nature of the formulæ, § VIII, that in the supposed nature of the coefficients $\overset{1}{A}$, $\overset{2}{A}$, &c. the numerical expression (μ) generally contains no higher power of y than y^μ.

2. Further, no expression of the form $(\alpha\beta\gamma\delta\ldots)$ can contain any higher power of y, than $y^{\alpha+\beta+\gamma+\delta}\ldots.$ The accuracy of this assertion appears from 1, together with the remark in § XXIV.

3. From § LXXV the first part of the final equation (the other$=$o) is the product of the m factors

$$x'^n+\overset{1}{A'}x'^{n-1}+\ldots\ldots+\overset{n-\mu}{A'}x'^\mu+\ldots\ldots+\overset{n}{A'}$$
$$x''^n+\overset{1}{A'}x''^{n-1}+\ldots\ldots+\overset{n-\nu}{A'}x''^\nu+\ldots\ldots+\overset{n}{A}$$
$$x'''^n+\overset{1}{A'}x'''^{n-1}+\ldots\ldots+\overset{n-\pi}{A'}x'''^\pi+\ldots\ldots+\overset{n}{A'}$$

&c.

after we have eliminated in it the roots x', x'', x''', &c. by means of the coefficients of the equation I.

4. The general term of this product is

$$\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}, \text{ \&c. } x'^\mu x''^\nu x'''^\pi\ldots\ldots$$

Now, since the product must be a symmetrical function of x', x'', x''', &c. this term necessarily belongs to

$$\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}\ \&c.\ [\mu\,\nu\,\pi\ldots\ldots]$$

5. But by 2, the highest exponent of y in $[\mu\nu\pi\ldots]$ is equal to the sum of the radical exponents $\mu+\nu+\pi+\ldots$; further, by the hypothesis, $n-\mu$ is the highest exponent of y in $\overset{n-\mu}{A'}$, $n-\nu$ the highest in $\overset{n-\nu}{A'}$, $n-\pi$ the highest in $\overset{n-\pi}{A'}$, &c. $\therefore\ mn-(\mu+\nu+\pi+\&c.)$ the highest in the product $\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}$ &c. But from both it follows, that mn is the highest exponent of y in the term $\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}$ &c. $[\mu\nu\pi\ldots]$, and that consequently in this term there can be no higher power of y than y^{mn}.

6. Now, since what has been here proved of an indeterminate term, obtains for every term in particular, it follows $\therefore$ that in the final equation there can be no higher power of y than y^{mn}.

SECTION LXXX.

When more than two equations with more than two unknown magnitudes are given, then in general there is no other way but to combine these equations in the usual way, two and two, and thus get rid of one unknown magnitude after the other. But Bezout observes, with truth, in the above-mentioned work, that this method is very defective, because a number of useless factors enters into the successive eliminations, by which not only the operation is lengthened, but likewise the degree of the final

equation becomes much higher than it ought to be, and what is much more objectionable is, that these factors do not show themselves till the calculation is completed. Since, however, these difficulties can be got rid of even by Bezout's method in no other way than by considering a great number of single cases, (but neither the object nor the limits of this work allow of such a detail as this), I shall consequently not enter into these inquiries at present, but leave them for consideration till a future period.

Tschirnhausen, in the "Acta Eruditorum," for the year 1683, has given a method for solving those equations which are founded solely on eliminations. This method only requires to transform the given equation, by means of an assumed auxiliary one, into another, which contains any number of indeterminate magnitudes, by the proper determination of which it is possible to remove as many terms as we please, and by that means give it the form of an equation of two terms, of a quadratic, of a cubic, or of any other equation, whose solution is already known, or may be considered as known. Its inventor considered it as general, and so it is indeed; only its application often requires the solution of higher equations than the given one itself. The following problems will elucidate what has been just said.

SECTION LXXXI.

PROB. Let the two equations

I. $x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \&c. = 0$

II. $y + A + Bx + Cx^2 + Dx^3 + \&c. = 0$

be given, in which the coefficients a, b, c, &c. A, B, C, &c.

contain neither x nor y: determine the degree of the final equation, which is obtained by the elimination of x, in terms of y.

Solution. Let x', x'', x''', &c. be the roots of the equation I; then, according to § LXXV, the first part of the final equation (the other $= 0$) is the product of the following m factors:

$$y + A + Bx' + Cx'^2 + Dx'^3 + \&c.$$
$$y + A + Bx'' + Cx''^2 + Dx''^3 + \&c.$$
$$y + A + Bx''' + Cx'''^2 + Dx'''^3 + \&c.$$
&c.

Now, since in these factors y occurs in no other term but the first, consequently in the product there can be no higher power of y than y^m. The equation in terms of y is ∴ necessarily of the mth degree, and consequently always of the same degree as the equation I, of which degree besides the equation II may also be.

Corollary. If ∴ an equation

$$x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \&c. = 0$$

be given, we can transform it into another of the same degree in numberless ways. For this purpose, we only need assume any auxiliary equation of the form

$$y + A + Bx + Cx^2 + Dx^3 + \&c.$$

and eliminate x from both equations. Now, since both the degree and the coefficients of the auxiliary equation are undetermined, we can always determine both, in the way required, by the form which we have determined on giving to the transformed equation. Thus, if we wish to

these values in the expression for K gives K, and at the same time the reduced equation $y^3 + K = o$.

Corollary. Having found the reduced equation, we are also enabled to find the roots of the given equation. Thus, from $y^3 + K = o$, we obtain, when $1, a, \beta$, denote the cube roots of unity, $y = -\sqrt[3]{K}$, $y = -a\sqrt[3]{K}$, $y = -\beta\sqrt[3]{K}$. If these values of y, together with those of A and B, be substituted in the auxiliary equation, we then obtain the required value of x by the solution of equations of the second degree.

Remark. Since the values of A and B depend on equations of the second degree, strictly speaking, we get 2×2 corresponding values of these magnitudes. Now, since every two corresponding values may be combined with each of the three values of y, these substitutions give six different equations of the second degree. Every one of these gives two values of x, and consequently we obtain generally twelve values of x, although the given equation can have no more than three roots.

We must, however, keep in mind, that only those values of x may be assumed, which verify at the same time the two equations I, II. In order to find these values, we need only $\therefore$ seek the common divisor of $x^3 + ax^2 + bx + c$ and $x^2 + Bx + A + y$ in the usual way. The division of the first expression by the second, gives the quotient $x + a - B$, and the remainder

$$(B^2 - aB + b - A - y)\,x + (B - a)\,(A + y) + c.$$

This remainder must vanish. We have consequently

$$(B^2 - aB + b - A - y)\,x + (B-a)\,(A+y)+c=0$$

and hence we obtain

$$x = -\frac{(B-a)\,(A+y)+c}{B^2 - aB + b - A - y}$$

If in this equation we substitute for A, B, their values from the equations in 4, also for y its three values successively, $-\sqrt[3]{K}, -\alpha\sqrt[3]{K}, -\beta\sqrt[3]{K}$, we obtain three values of x, which are the roots of the given equation. Moreover, it matters not, in this case, which we use of the corresponding values of A and B, because they always give the same values of x; of which we can easily persuade ourselves by actual calculation.

SECTION LXXXII.

PROB. Transform the general equation of the fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

into another of the form $y^4 + Hy^2 + K = 0$.

Solution 1. Since the transformed equation is to have the form $y^4 + Hy^2 + K = 0$, in which two terms vanish, viz. the second and fourth, we must consequently assume an auxiliary equation, with two arbitrary magnitudes. Let ∴

$$y + A + Bx + x^2 = 0$$

be this auxiliary equation.

2. In order from

$$\text{I.}\quad y + A + Bx + x^2 = 0$$

$$\text{II.}\quad y^4 + ax^3 + bx^2 + cx + d = 0$$

to eliminate x, it is only necessary to compare these equations with those of the first example in § LXXVII; we then find $A'=0$, $B'=1$, $C'=a$, $D'=b$, $E'=c$, $F'=d$, $A=-B$, $B=y+A$. If we make these substitutions in the final equation in the above mentioned §, it is by these means transformed into an equation of the form

$$(y+A)^4+P(y+A)^3+Q(y+A)^2+R(y+A)+S=0$$

and then

$$\begin{aligned}
P &= -aB + a^2 - 2b \\
Q &= bB^2 + (3c-ab)B + b^2 - 2ac + 2d \\
R &= -cB^3 + (ac-4d)B^2 + (3ad-bc)B \\
&\qquad + c^2 - 2bd \\
S &= dB^4 - adB^3 + bdB^2 - cdB + d^2
\end{aligned}$$

3. If we arrange this equation according to y, we obtain

$$\begin{aligned}
& y^4 + (4A + P)y^3 + (6A^2 + 3PA + Q)y^2 \\
& \quad + (4A^3 + 3PA^2 + 2QA + R)y \\
& \quad + A^4 + PA^3 + QA^2 + RA + S = 0
\end{aligned}$$

in which any two terms may be eliminated at pleasure, by merely determining the letters A, B, conformably to it.

4. Now, in order, as the problem requires, to eliminate the second and fourth terms, we put

$$\begin{aligned}
& 4A + P = 0 \\
& 4A^3 + 3PA^2 + 2QA + R = 0
\end{aligned}$$

5. The first gives $A = -\frac{P}{4}$, and if we introduce this

value into the other equation, by omitting the fractions, we have

$$p^3 - 4PQ + 8R = 0$$

If in this equation we make use of the above values of P, Q, R, we then obtain for B an equation of the third degree, viz.

$$(4ab-a^3-8c)B^3+(3a^4-14a^2b+20ac+8b^2-32d)B^2$$
$$+(-3a^5+16a^3b-16ab^2-20a^2c+32ad+16bc)B$$
$$+a^6-6a^4b+8a^3c-8a^2d+8a^2b^2-16abc+8c^2$$
$$=0$$

Having determined B from this equation, we need only substitute its value in the above expressions for P, Q, R, S, in order to find these coefficients also.

6. Further, the equation in 3, by putting $-\frac{P}{4}$ for A is transformed into

$$y^4 - (\frac{3P^2}{8} - Q)y^2 - \frac{3P^4}{256} + \frac{QP^2}{16} - \frac{RP}{4} + S = 0$$

or, when we substitute for R its values $\frac{PQ}{2} - \frac{P^3}{8}$ into

$$y^4 - (\frac{3P^2}{8} - Q)y^2 + \frac{5P^4}{256} - \frac{QP^2}{16} + S = 0;$$

and this equation has the form $y^4 + Hy^2 + K = 0$, as required.

Corollary. Now from this transformed equation we may find the roots of the equation $x^4+ax^3+bx^2+cx+d=0$ in a similar way as in the foregoing § for the equation of the third degree. Thus, since the equation in 5 gives

three values for B, and the substitution of each of these values in the transformed equation gives four values of y, we $\therefore$ obtain generally twelve values for y. Each of these values of y, together with that of B when substituted in the auxiliary equation $x^2 + Bx + A + y = 0$, or $x^2 + Bx - \frac{P}{4} + y = 0$, gives two values for x, and we $\therefore$ obtain generally twenty-four values of x. Now, in order to learn which of these values are at the same time roots of the given equation, we must seek the common divisors of the two expressions $x^4 + ax^3 + bx^2 + cx + d$, $x^2 + Bx - \frac{P}{4} + y$. With this view, we divide the first expression by the last, until we come to a remainder, which contains x in the first power only; this remainder must $\therefore$ be $= 0$. In this way we obtain the equation

$$[B^3 - aB^2 + bB - c - (a - 2B)(y - \frac{P}{4})]\,x$$
$$+ (B^2 - aB + b)(y - \frac{P}{4}) - (y - \frac{P}{4})^2 - d = 0$$

and hence

$$x = \frac{d - (B^2 - aB + b)(y - \frac{P}{4}) + (y - \frac{P}{4})^2}{B^3 - aB^2 + bB - c - (a - 2B)(y - \frac{P}{4})}$$

Now, if we substitute in this expression of x for y its four values from the transformed equation, we thus obtain the roots of the given equation, and indeed we shall always find the same four values for x, whichever value of B we make use of.

REMARK. From this and the two foregoing sections we deduce at least this much, that Tschirnhausen's method leads to the actual solution of equations of the second, third, and fourth degree, although in a very laborious way. Whether, and how far this method is also applicable to higher degrees, will be the subject of inquiry hereafter.

V.—On the roots of the equation $x^n - 1 = 0$, and its application to the elimination of surds from equations. A method, by which to find solvable equations, and some other subjects connected with it.

SECTION LXXXIV.

Prob. Find an equation, which merely contains the imaginary roots of the equation $x^n - 1 = 0$.

Solution. Here two cases must be taken into consideration, viz. first, when n is an odd; secondly, when n is an even number.

1. Let n be an odd number. In this case there is no more than one real root, viz. $+ 1$, and consequently $x - 1$ must be a divisor of $x^n - 1$. If $\therefore$ we divide the equation $x^n - 1 = 0$ by $x - 1$, and make the quotient $= 0$, we obtain an equation which only contains the imaginary roots, and this is

$$x^{n-1} + x^{n-2} + x^{n-3} + \ldots\ldots + x^2 + x + 1 = 0$$

2. Let n be an even number, $\therefore$ the given equation is of the form $x^{2m} - 1 = 0$. In this case it has necessarily two real roots, viz. $+ 1$ and $- 1$, and no more. Consequently both $x - 1$ and $x + 1$ must be divisors of x^{n-1},

$\therefore$ also the product $(x-1)\ (x+1)=x^2-1$. If $\therefore$ we divide the equation $x^n-1=o$ by x^2-1, we thus obtain an equation, which only contains the imaginary roots, and this is

$$x^{n-2} + x^{n-4} + x^{n-6} + \ldots + x^4 + x^2 + 1 = o$$

Corollary. In order $\therefore$ to find all the roots of the equation $x^n-1=o$, we must, when n is an odd number, endeavour to solve the equation $x^{n-1} + x^{n-2} + \ldots + x + 1=o$, and when n is an even number, the equation $x^{n-2}+x^{n-4}+\ldots+x^2+1=o$. The latter, because it only contains even powers of x, may always be reduced to an equation of the degree $\frac{n-2}{2}$ by substituting y for x^2.

EXAMPLE I. The equation $x^3-1=o$ divided by $x-1$, gives

$$x^2 + x + 1 = o$$

and when solved, $x = \frac{-1\pm\sqrt{-3}}{2}$. The three roots of this equation are consequently

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

EXAMPLE II. The equation $x^4-1=o$ divided by x^2-1, gives

$$x^2 + 1 = o$$

whence we obtain $x=\pm\sqrt{-1}$. The four roots of the equation $x^4-1=o$ are consequently

$$+1,\ -1,\ +\sqrt{-1},\ -\sqrt{-1}$$

EXAMPLE III. The equation $x^5-1=0$ divided by $x-1$, gives

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

This equation may be analyzed into two quadratic equations

$$x^2 + (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})\,x + 1 = 0$$
$$x^2 + (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})\,x + 1 = 0$$

and the solution of these two equations gives the four following imaginary rooots:

$$\tfrac{1}{4}[-1-\sqrt{5} + \sqrt{(10-2\sqrt{5})}\sqrt{-1}]$$
$$\tfrac{1}{4}[-1-\sqrt{5} - \sqrt{(10-2\sqrt{5})}\sqrt{-1}]$$
$$\tfrac{1}{4}[-1+\sqrt{5} + \sqrt{(10+2\sqrt{5})}\sqrt{-1}]$$
$$\tfrac{1}{4}[-1+\sqrt{5} - \sqrt{(10+2\sqrt{5})}\sqrt{-1}]$$

EXAMPLE IV. The equation $x^6-1=0$ divided by x^2-1, gives

$$x^4 + x^2 + 1 = 0$$

and the solution of this equation gives

$$x = \pm\sqrt{\frac{-1\pm\sqrt{-3}}{2}}$$

The six roots of the equation $x^6-1=0$ are $\therefore$

$$+1 \qquad -1$$
$$+\sqrt{\frac{-1+\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1+\sqrt{-3}}{2}}$$
$$+\sqrt{\frac{-1-\sqrt{-3}}{2}}, \quad -\sqrt{\frac{-1-\sqrt{-3}}{2}}$$

SECTION LXXXV.

PROB. Reduce the equation $x^n-k=0$ to an equation of the form $y^n-1=0$.

Solution. Put $x=y\sqrt[n]{k}$, and substitute this value in the equation $x^n-k=0$, then this equation is transformed into $ky^n-k=0$, or when divided by k, into $y^n-1=0$.

Corollary. If ∴ we have in any way already solved the equation $y^n-1=0$, and denote by $1, \alpha, \beta, \gamma, \delta, \epsilon$, &c. its n roots, or the value of y, we then obtain from $x=y\sqrt[n]{k}$ the n following roots of the equation $x^n-k=0$:

$$\sqrt[n]{k},\ \alpha\sqrt[n]{k},\ \beta\sqrt[n]{k},\ \gamma\sqrt[n]{k},\ \delta\sqrt[n]{k},\ \epsilon\sqrt[n]{k},\ \text{\&c.}$$

SECTION LXXXVI.

Prob. Reduce the equation $x^{pq}-1=0$ to an equation of the form $y^q-1=0$.

Solution. Put $x^p=y$, then $x^{pq}=y^q$. If this value be substituted in the given equation, it is transformed into $y^q-1=0$.

Corollary. If we denote the roots of the equation $x^p-1=0$ by $1, \alpha, \beta, \gamma, \delta, \epsilon$, &c., then the roots of the equation $x^p-y=0$ (foregoing §) are

$$\sqrt[p]{y},\ \alpha\sqrt[p]{y},\ \beta\sqrt[p]{y},\ \gamma\sqrt[p]{y},\ \delta\sqrt[p]{y},\ \epsilon\sqrt[p]{y},\ \text{\&c.}$$

Now, since we can substitute for y each of the roots of the equation $y^q-1=0$, we obtain by these substitutions all the pq roots, of the equation $x^{pq}-1=0$.

Example. To find the roots of the equation $x^{12}-1=0$, put $p=4$, $p=3$. We consequently have the two equations

$$x^4 - y = 0, \quad y^3 - 1 = 0.$$

Now, the roots of the equation $x^4 - 1 = 0$ (§ LXXXIV, example II). are $+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, $\therefore$ the roots of the equation $x^4 - y = 0$ (foregoing §).

$$\sqrt[4]{y},\ -\sqrt[4]{y},\ +\sqrt{-1}\,.\,\sqrt[4]{y},\ -\sqrt{-1}\,.\,\sqrt[4]{y}.$$

Further, the roots of the equation $y^3 - 1 = 0$ (§ LXXXIV, example I).

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}$$

If we successively substitute these values for y, we obtain the twelve following roots of the equation $x^{12} - 1 = 0$:

$$1, \qquad -1, \qquad \sqrt{-1}, \qquad -\sqrt{-1}$$

$$\sqrt[4]{\frac{-1+\sqrt{-3}}{2}},\ -\sqrt[4]{\frac{-1+\sqrt{-3}}{2}},\ \sqrt{-1}\,.\,\sqrt[4]{\frac{-1+\sqrt{-3}}{2}},\ -\sqrt{-1}\,.\,\sqrt[4]{\frac{-1+\sqrt{-3}}{2}},$$

$$\sqrt[4]{\frac{-1-\sqrt{-3}}{2}},\ -\sqrt[4]{\frac{-1-\sqrt{-3}}{2}},\ \sqrt{-1}\,.\,\sqrt[4]{\frac{-1-\sqrt{-3}}{2}},\ -\sqrt{-1}\,.\,\sqrt[4]{\frac{-1-\sqrt{-3}}{2}}.$$

SECTION LXXXVII.

Prob. Under the supposition that n is a prime number, from any one of the imaginary roots of the equation $x^n - 1 = 0$, find all the remaining ones.

Solution 1. Let a denote one of the imaginary roots of the equation $x^n - 1 = 0$, so that $a^n - 1 = 0$, or $a^n = 1$.

2. Since $a^n = 1$, then also $(a^m)^n = (a^n)^m = 1$. If $\therefore$ a is a root of the equation $x^n - 1 = 0$, then must also

a^m be one of its roots. Therefore the equation $x^n - 1 = o$ has, besides a, the roots a^2, a^3, a^4, a^5, &c.

3. But since in this way we should find an infinite number of roots, and the equation $x^n - 1 = o$ can only have n roots, we may safely presume, that in the series of powers a, a^2, a^3, a^4, a^5, &c. there must be an infinite number of equal roots. This likewise is really the case: for we find $a^{n+1} = a^n \cdot a = a$, $a^{n+2} = a^n \cdot a^2 = a^2$, $a^{n+3} = a^n \cdot a^3 = a^3$, &c.

4. Generally, when we exceed the nth power, we shall find only one of the n following roots

$$a,\ a^2,\ a^3,\ a^4,\ a^5,\ \ldots\ldots\ldots\ a^{n-1},\ a^n$$

of which the last $= 1$. For let a^m be any power of a, and $m > n$. Further, let q denote the quotient, which we obtain after dividing m by n, and r the remainder, consequently $r < n$; then $m = nq + r$. We have $\therefore$ $a^m = a^{nq+r} = a^{nq} \cdot a^r = (a^n)^q \cdot a^r = 1^q \cdot a^r = a^r$. But a^r, since $r < n$, is necessarily one of the powers a, a^2, a^3, a^4, ... a^{n-1}.

5. The conclusions drawn hitherto obtain, whether n be a prime number or not. In the particular case, when n is a prime number, according to the supposition in the problem, it may be proved, that the roots a, a^2, a^3, ... a^n, are all different from one another. For we suppose two of these roots a^μ, a^ν, to be equal, and $\nu > \mu$. Then we divide the equation $a^\nu = a^\mu$ by a^μ, and obtain $a^{\nu-\mu} = 1$; but it may be shown, as follows, that this equation is impossible.

6. Thus, since n is a prime number, and $\nu - \mu < n$, the numbers $\nu - \mu$ and n are $\therefore$ prime to one another. Consequently, as is already known from indeterminate analysis, two whole positive numbers t, u, may always be found, such that $(\nu - \mu)\,t = nu + 1$. If $\therefore$ $a^{\nu-\mu} = 1$, then also must $a^{(\nu-\mu)t} = 1$, and consequently also $a^{nu+1} = 1$, or $a^{nu} \cdot a = 1$, or $a = 1$; which is impossible (1).

7. Since $\therefore$ the roots $a, a^2, a^3, a^4, \ldots\ldots a^n$, as far as the number n, are all different from one another, then these are the n roots of the equation $x^n - 1 = 0$. If $\therefore$, an imaginary root a be given, we then have likewise all the remaining ones.

Corollary. If $\therefore$ we denote the imaginary roots of the equation $x^n - 1 = 0$ by a, β, γ, δ, &c., then, when n is a prime number, all the roots of this equation may be represented in one or other of the following ways:

either by $a,\ a^2,\ a^3 \ldots\ldots\ldots a^{n-1},\ a^n$

or by $\beta,\ \beta^2,\ \beta^3 \ldots\ldots\ldots \beta^{n-1},\ \beta^n$

or by $\gamma,\ \gamma^2,\ \gamma^3 \ldots\ldots\ldots \gamma^{n-1},\ \gamma^n$

&c.

or, which is the same, we can substitute in the series of roots $a, a^2, a^3, \ldots\ldots a^{n-1}, a^n$, for a each imaginary root $a^2, a^3, a^4 \ldots\ldots a^{n-1}$, and then we shall always obtain the same n roots.

Example. When $n = 3$, the roots are a, a^2, a^3. If for the root a, we substitute the following one a^2, we

obtain a^2, a^4, a^6. But since $a^3 = 1$, then $a^4 = a$, and $a^6 = a^3$, and we have $\therefore$ here a^2, a, a^3, as before. When $n = 5$, the roots are a, a^2, a^3, a^4, a^5. If we put a^2 for a, then, on the contrary, we have a^2, a^4, a^6, a^8, a^{10}, or, since $a^5 = 1$, a^2, a^4, a, a^3, a^5; consequently the same roots as before. In like manner, when a^3 is put for a, we find a^3, a^6, a^9, a^{12}, a^{15}, or a^3, a, a^4, a^2, a^5, and when a^4 is put for a, a^4, a^8, a^{12}, a^{16}, a^{20}, or a^4, a^3, a^2, a, a^5; consequently always the same roots, only in a different order.

SECTION LXXXVIII.

Rules.

I. When n is divisible by m, then all the roots of the equation $x^m - 1 = 0$, must also be roots of the equation $x^n - 1 = 0$.

Proof. Since n is divisible by m, then $\frac{n}{m} = q$ is a whole number, and $n = qm$. The equation $x^n - 1 = 0$ is consequently $x^{qm} - 1 = 0$, and if we put $x^m = y$, $y^q - 1 = 0$. Now, $y^q - 1 = 0$ is divisible by $y - 1$; consequently also, if again x^m is put for y, $x^{qm} - 1$ is divisible by $x^m - 1$; $\therefore$ the roots of the equation $x^m - 1 = 0$ are also roots of the equation $x^{qm} - 1 = 0$, or $x^n - 1 = 0$. Q. E. D.

II. When a root (unity excepted) of the equation $x^n - 1 = 0$, which is also a root of the equation $x^m - 1 = 0$, is of a low degree, and that of the very lowest possible, then n must be divisible by m.

Proof. Let a be the common root, consequently $a^n - 1 = o$, and $a^m - 1 = o$. Now, if n be not divisible by m, then n divided by m gives the quotient q and the remainder r, so that $n = qm + r$, and $r < m$. Then $a^n = a^{qm+r} = a^{qm} . a^r$. But $a^n = 1$, $a^{qm} = (a^m)^q = 1$; $\therefore 1 = a^r$; consequently a is also a root of the equation $x^r - 1 = o$, which is contrary to the hypothesis, that $x^m - 1 = o$ is the lowest equation, which contains the root a.

III. When m, n are two numbers, which have no common measure, then the equations $x^m - 1 = o$, $x^n - 1 = o$ have no common root, except unity.

Proof. If possible, let the two equations have a common root a, different from unity, then, at the same time, $a^m = 1$ and $a^n = 1$. Since m and n are prime to each other, we can always find two whole positive numbers, t, u, such, that $mt = nu + 1$. We have then the equation $a^{mt} = a^{nu+1} = a^{nu} . a$. But according to the hypothesis $a^m = a^n = 1$, consequently $a^{mt} = a^{nu}$: $\therefore 1 = a$; which is contrary to the hypothesis.

IV. When the two equations $x^n - 1 = o$, $x^m - 1 = o$, have another root besides unity, common to both, then the exponents m, n, must have a common measure.

Proof. For if they have no common measure, then also they can have no common root except unity (III).

SECTION LXXXIX.

PROB. Let the equation $x^n - 1 = 0$ be given, and n be a compound number: find all those roots of this equation, which do not belong to any equation of a lower degree of the same form.

Solution 1. Let p, q, r, &c. be the simple factors of the exponent n; further, let $\frac{n}{p} = \mu, \frac{n}{q} = \mu', \frac{n}{r} = \mu''$&c. consequently n is divisible by μ, μ', μ'', &c.

2. Construct ∴ the equations

$$x^\mu - 1 = 0,\ x^{\mu'} - 1 = 0,\ x^{\mu''} - 1 = 0,\ \&c.$$

then these equations must have all their roots in common with the equation $x^n - 1 = 0$ (§ LXXXVIII, I.)

3. Now, I affirm, that each root of the equation $x^n - 1 = 0$, which also belongs to a lower equation of this form, must necessarily be a root of one of the equations in 2. For, let a be a common root of the equations $x^n - 1 = 0$, $x^k - 1 = 0$, and the last the lowest of this form, to which this root can belong, then k must be a divisor of n (§ LXXXVIII, II.), consequently also assuredly a divisor of one of the numbers μ, μ', μ'', &c. Therefore, all the roots of $x^k - 1 = 0$ must be contained in one of the equations in 2; and consequently also the root a.

4. If we divide the equation $x^n - 1 = 0$ successively by $x^\mu - 1$, $x^{\mu'} - 1$, $x^{\mu''} - 1$, &c., we obtain

$$x^{n-\mu} + x^{n-2\mu} + x^{n-3\mu} + \ldots. + x^{2\mu} + x^{\mu} + 1 = 0$$
$$x^{n-\mu'} + x^{n-2\mu'} + x^{n-3\mu'} + \ldots. + x^{2\mu'} + x^{\mu'} + 1 = 0$$
$$x^{n-\mu''} + x^{n-2\mu''} + x^{n-3\mu''} + \ldots. + x^{2\mu''} + x^{\mu''} + 1 = 0$$
&c.

The first of these equations contains all those roots of $x^n - 1 = 0$, which are not contained in $x^{\mu} - 1 = 0$; the second all those roots of $x^n - 1 = 0$, which are not contained in $x^{\mu'} - 1 = 0$; &c.

5. A root β, which is common to all these equations, cannot be found in any one of the equations $x^{\mu} - 1 = 0$, $x^{\mu'} - 1 = 0$, $x^{\mu''} - 1 = 0$, &c. and $\therefore$ cannot be a root of an equation of two terms of a lower degree than $x^n - 1 = 0$ (3).

6. If $\therefore$ we seek the greatest common divisor of the equations in 4, then this must contain only such roots as are peculiar to the equation $x^{\mu} - 1 = 0$, and belong to none of a lower degree of this form. But it is also evident, that there can be no such root wanting in the greatest common divisor, because otherwise it could not be the greatest.

Example I. Let $x^4 - 1 = 0$ be the given equation, consequently $n = 4$. Since this number has only one simple factor, viz 2, $\therefore p = 2$; consequently $\mu = \frac{n}{p} = 2$. If $\therefore$ we divide the equation $x^4 - 1 = 0$ by $x^2 - 1$, we obtain the equation

$$x^2 + 1 = 0$$

whose roots $+\sqrt{-1}$ and $-\sqrt{-1}$ are such, that they do not become $+1$ till raised to the fourth power.

Example II. Let $x^{12}-1=0$ be the given equation, consequently $n = 12$. This number has two simple factors, viz. 2 and 3. We have $\therefore p=2$, $q=3$, and consequently $\mu = \frac{n}{p} = 6$, $\mu' = \frac{n}{q} = 4$. Now, if we divide $x^{12}-1=0$ by x^6-1 and x^4-1, we obtain the two equations

$$x^6 + 1 = 0$$
$$x^8 + x^4 + 1 = 0.$$

Their greatest common divisor is

$$x^4 - x^2 + 1 = 0.$$

Hence we find

$$x = \pm\sqrt{\frac{1 \pm \sqrt{-3}}{2}} = \pm\frac{\sqrt{3} \pm \sqrt{-1}}{2}$$

and these four roots are peculiar to the equation $x^{12}-1=0$, because they do not become $+1$ till raised to the twelfth power.

In order to find the above roots, it is only necessary to solve the equations $x^6-1=0$, $x^4-1=0$, and to take the common roots only once. The roots of the equations $x^4-1=0$, $x^6-1=0$ are in § LXXXIV. In this way we obtain the following eight roots.

$$\pm 1,\ \pm\sqrt{-1},\ \pm\sqrt{\frac{-1+\sqrt{-3}}{2}},\ \frac{\pm\sqrt{-1-\sqrt{-3}}}{2}$$

which together with the four preceding, give the twelve roots of $x^{12}-1=0$. This mode of expressing them, is, as we see, much more simple than that in § LXXXVI.

Remark. A root which is peculiar to the equation $x^n-1=0$, and which consequently belongs to no equation of

a lower degree of this form, is termed a primitive root of this equation.

SECTION XC.

PROB. Let n be a compound number, and a a given primitive root: find all the roots of this equation.

Solution 1. In § LXXXVII it has been proved, that for every n, though a may be any imaginary root, the powers a, a^2, $a^3 \ldots a^n$, are always roots of the equation $x^n - 1 = 0$.

2. I affirm, then, that when, as has been here supposed, a is a primitive root, in the series of magnitudes a, a^2, $a^3, \ldots \ldots a^n$, there are no two, which are equal to one another. For if $a^\mu = a^\nu$, then $a^{\mu-\nu} = 1$, consequently a is a root of the equation $x^{\mu-\nu} - 1 = 0$; therefore the root of an equation of the form $x^n - 1 = 0$, of a lower degree than n, and consequently no primitive root, which is contrary to the hypothesis.

3. Since ∴ the magnitudes a, a^2, a^3, $a^4, \ldots \ldots a^n$ are all roots of the equation $x^n - 1 = 0$, and all different from one another, they are the n roots of this equation, which were sought.

SECTION XCI.

PROB. Let n be a compound number, and a a primitive root of the equation $x^n - 1 = 0$, ∴ a, a^2, a^3, $a^4, \ldots a^n$,

all the roots of this equation (§ XC): find a criterion by which to distinguish the primitive roots of this equation from the others.

Solution 1. If two whole numbers m, n, have a common measure, there may always be found a whole number t, which is less than n, and such, that mt is divisible by n; on the other hand, if the numbers m, n, are prime to each other, then t cannot be less than n, if mt is divisible by n.

2. Now let a^m be any one of the magnitudes a, a^2, a^3, ... a^n. If this be a root of an equation $x^t - 1 = 0$, we then must have $a^{mt} - 1 = 0$, or $a^{mt} = 1$; $\therefore$ mt must be divisible by n.

3. From this condition and from 1 it follows, that when the numbers m, n, have a common measure, there can always be found an equation $x^t - 1 = 0$ of a lower degree than the nth, of which a^m is a root; but that no such equation can be found, when m, n, are prime to each other.

4. But hence it follows, that of the powers a, a^2, a^3, a^4, ... a^n, all those, without exception, are primitive roots of the equation $x^n - 1 = 0$, whose exponents have no common measure with n; and this $\therefore$ is the criterion by which the primitive numbers may be distinguished from the others.

EXAMPLE. Amongst all the roots a, a^2, a^3, a^4, a^5, a^6

a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, of the equation $x^{12} - 1 = 0$, there are no more than the four a, a^5, a^7, a^{11}, whose exponents have no common measure with 12, and consequently which are primitive roots of this equation; and these roots can be no other than the four which were found in the second example, § LXXXIX; viz.

$$\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1}\,,\quad \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1}$$
$$-\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1}\,,\ -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1}$$

In order to be convinced of this, assume one of them, viz. $\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1}$, for a; by actually raising this root to the fifth, seventh, and eleventh powers, we find:

$$\begin{aligned} a &= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1} \\ a^5 &= -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1} \\ a^7 &= -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1} \\ a^{11} &= \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1} \end{aligned}$$

and these are the same as the foregoing. We should have obtained the same result, if we had put every other of the four above-mentioned roots for a. That this must be the case, may, besides, be seen without actually completing the calculation; for if in a, a^5, a^7, a^{11}, we substitute a^5, a^7, a^{11}, successively for a, and omit in the exponents the multiples of 12, we then obtain

$$\begin{array}{llll} a^5,\ a^{25},\ a^{35},\ a^{55}, & \text{or } a^5,\ a,\ a^{11},\ a^7 \\ a^7,\ a^{35},\ a^{49},\ a^{77}, & \text{or } a^7,\ a^{11},\ a,\ a^5 \\ a^{11},\ a^{55},\ a^{77},\ a^{121}, & \text{or } a^{11},\ a^7,\ a^5,\ a, \end{array}$$

∴ always the same roots, only in a different order.

SECTION XCII.

If we compare the equation $x^n - 1 = 0$ with the general equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \ldots\ldots$

$\mp Sx \pm T = o$, $A = o$ $B = o$, $C = o$, ... $S = o$, $T = \mp 1$. If $\therefore$ we denote the roots of this equation by $\alpha, \beta, \gamma, \delta$, &c. we have

$$\alpha + \beta + \gamma + \delta + \&c. = o$$
$$\alpha\beta + \alpha\gamma + \&c. + \beta\gamma + \beta\delta + \&c.\&c. = o$$
$$\alpha\beta\gamma + \alpha\beta\delta + \&c. + \beta\gamma\delta + \&c.\&c. = o$$

and so on to the product of all the roots, which is $= -1$, or $= +1$, according as n is even or odd.

Since in the two first chapters, the letters, $\alpha, \beta, \gamma, \delta$, &c. are used to denote the radical exponents, in order to prevent mistakes, I shall once for all remind my readers, that these letters, when they are in the brackets [ ], always denote, as heretofore, radical exponents, but in every other case the roots themselves. Further, in order to indicate, that a numerical expression relates exclusively to the roots of an equation of the form $x^n - 1 = o$, I shall place a dash over the left side of the bracket thus $`[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ in reference to the equation $x^n - 1 = o$, denotes a numerical expression for the radical exponents $\alpha, \beta, \gamma, \delta, \ldots\ldots \kappa$.

SECTION XCIII.

Prob. Find the sums of the powers of the roots of the equation $x^n - 1 = o$.

Solution 1. If we compare this equation with the general one

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Px + Q = o,$$

we find $A = o$, $B = o$, $C = o$, &c.; $P = o$, $Q = -1$. Consequently, by means of the equations in 7, § VIII, we obtain

$$'[1] = 0,\ '[2] = 0,\ '[3] = 0, \ldots\ldots '[n-1] = 0$$

on the other hand $'[n] = n$. In like manner we find

$$'[n+1]=0,\ '[n+2]=0,\ '[n+3]=0, \ldots\ldots '[2n-1]=0$$

on the other hand $'[\overline{2n}]=n$. Generally, all those sums of powers, whose radical exponents are divisible by n, are equal to n, all the remaining ones, on the contrary, $=0$.

2. If we put $x = \frac{1}{y}$, the equation $x^n - 1 = 0$, is transformed into $\frac{1}{y^n} - 1 = 0$, or $y^n - 1 = 0$, whose roots consequently are the reciprocals of the roots of the former equation (§ X). But since the equations $x^n - 1 = 0$, $y^n - 1 = 0$, are similar to one another, we also have

$$'[-1]=0,\ '[-3]=0,\ \ldots\ldots\ '[-n+1]=0,\ '[-n]=n$$

and generally, all those negative sums of powers, whose exponents are divisible by n, $=n$, all the remaining ones, on the contrary, $=0$.

SECTION XCIV.

Prob. Find the value of $'[\alpha\beta]$.

Solution. Since, generally, for every equation $[\alpha\beta] = [\alpha]\ [\beta] - [\alpha + \beta]$, and when $\alpha = \beta$, $2[\alpha^2] = [\alpha]^2 - [\overline{2\alpha}]$, so also in particular for the equation $x^n - 1 = 0$.

$$'[\alpha\beta] = '[\alpha]\ '[\beta] - '[\alpha + \beta]$$
$$'[\alpha^2] = \tfrac{1}{2}\ '[\alpha]^2 - \tfrac{1}{2}\ '[\overline{2\alpha}]$$

In order to determine from hence the numerical value of $'[\alpha\beta]$, there must be three different cases.

1. When $\alpha+\beta$ is divided by n, but the radical exponents α, β, are not taken singly. In this case, according to the foregoing §, $`[\alpha] = `[\beta] = 0$, $`[\alpha + \beta] = n$; consequently

$$`[\alpha\beta] = -n,$$

and when $\beta = \alpha$

$$`[\alpha^2] = -\frac{n}{2}.$$

2. When $\alpha+\beta$ is divisible by n, but at the same time also, the radical exponents α, β, are taken singly. In this case $`[\alpha] = `[\beta] = n$, and $`[\alpha+\beta] = n$; $\therefore$

$$`[\alpha\beta] = n^2 - n$$

and when $\alpha = \beta$

$$`[\alpha^2] = \frac{n^2 - n}{2}.$$

3. When $\alpha+\beta$ is not divisible by n. In this case $`[\alpha + \beta] = 0$; but likewise the product $`[\alpha]\, `[\beta] = 0$, because then α and β cannot at the same time both be divisible by n; consequently always

$$`[\alpha\beta] = 0.$$

SECTION XCV.

Prob. Required to find the value of $`[\alpha\beta\gamma]$.

Solution. Now

$$`[\alpha\beta\gamma] = `[\alpha]\, `[\beta]\, `[\gamma] - `[\alpha]\, `[\beta+\gamma] - `[\beta]\, `[\alpha+\gamma] - `[\gamma]\, `[\alpha + \beta] + 1\,.\,2\, `[\alpha + \beta + \gamma].$$

Here, then, the following cases are to be considered:

1. When $\alpha + \beta + \gamma$ is not divisible by n, then the last term of the solution of $`[\alpha\beta\gamma]$ here given $= 0$; then also all the remaining terms of this solution $= 0$, because in each of these there must be at least one numerical expression with one radical exponent not divisible by $n$: consequently for this case $`[\alpha\beta\gamma] = 0$.

2. If each of the magnitudes α, β, γ, be divisible by n, then each of the numerical expressions $`[\alpha]$, $`[\beta]$, $`[\gamma]$, $`[\alpha + \beta]$, $`[\alpha + \gamma]$, $`[\beta + \gamma]$, $= n$, and consequently

$$`[\alpha\beta\gamma] = n^3 - 3n^2 + 2n.$$

3. In every other case always

$$`[\alpha\beta\gamma] = - n^2 + 2n.$$

SECTION XCVI.

PROB. Find the value of the general numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$.

Solution 1. The last term in the development of this numerical expression is, according to the second chapter,

$$\mp \frac{1 \, . \, 2 \, . \, 3 \, . \, 4 \ldots\ldots\ldots\ldots\ldots\ldots \nu - 1}{1 . 2 \ldots \mathfrak{a} \times 1 . 2 \ldots \mathfrak{b} \times \ldots \times 1 . 2 \ldots \mathfrak{k}} `(\mathfrak{a}\alpha + \mathfrak{b}\beta + \ldots + \mathfrak{k}\kappa)$$

when $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \ldots\ldots + \mathfrak{k}$ is put $= \nu$; the upper sign when ν is even, and the under one when ν is odd.

2. Now, if n be not a divisor of $\mathfrak{a}\alpha + \mathfrak{b}\beta + \mathfrak{c}\gamma + \ldots + \mathfrak{k}\kappa$, then this last term $= 0$; but at the same time also, all the remaining terms vanish, because in each of these there must be at least one numerical expression, whose radical exponent is not divisible by n, for otherwise the sum of all

the radical exponents, contrary to the supposition, must be divisible by n.

3. But if $\mathfrak{a}\alpha+\mathfrak{b}\beta+\mathfrak{c}\gamma+\ldots\ldots+\mathfrak{k}\kappa$, be divisible by n, then this last term alone is

$$\mp\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,\nu-1}{1\,.\,2\ldots\mathfrak{a}\times1\,.\,2\ldots\mathfrak{b}\times\ldots\ldots\times1\,.\,2\ldots\mathfrak{k}}.n$$

4. In order to obtain the values of the remaining terms, we can proceed as follows. Analyze the expression $\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}$ in all possible ways into combinations of two, three, and so on magnitudes, the sums of whose numbers are divisible by n, and assign to each such combination the coefficient K in 7, § XXX, when all the combinations are different; on the other hand, the coefficient K in 10, § XXX, when some of them are equal, and determine the sign as in 13. Then, without any further reference to the combinations themselves, take the aggregate of all the coefficients, which arise from the divisions into two combinations; further, the aggregate of all the coefficients which arise from the divisions into three combinations, and so on. If, then, we denote these different aggregates in their order by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, then the value of all the terms besides the last

$$=\mathfrak{A}n^2+\mathfrak{B}n^3+\mathfrak{C}n^4+\mathfrak{D}n^5+\&c.$$

5. From 3 and 4 we obtain $\therefore$ the following value of the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$:

$$\mp\frac{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\ldots\ldots\ldots\ldots\,\nu-1}{1\,.\,2\ldots\mathfrak{a}\times1\,.\,2\ldots\mathfrak{b}\times\ldots\ldots\times1\,.\,2\times\mathfrak{k}}.n$$
$$+\mathfrak{A}n^2+\mathfrak{B}n^3+\mathfrak{C}n^4+\mathfrak{D}n^5+\&c.$$

Example. Suppose it is wished to find the value of $`[1^2 2^3 5^2]$, when $n = 6$, we then proceed according to the following scheme:

Combination $1^2 2^3 5^2$

Divisions.	*Coefficients.*
$15, \quad 12^3 5$	$-\dfrac{1.2.3.4}{1.2.3} = -4$
$2^3, \quad 1^2 5^2$	$-\dfrac{1.2 \times 1.2.3}{1.2.3 \times 1.2 \times 1.2} = -\frac{1}{2}$
$1^2 2^2, \quad 25^2$	$-\dfrac{1.2.3 \times 1.2}{1.2 \times 1.2 \times 1.2} = -\frac{3}{2}$
$15, \quad 15, \quad 2^3$	$+\dfrac{1.2}{1.2 \times 1.2.3} = +\frac{1}{6}$

We have $\therefore\ \mathfrak{A} = -4 - \frac{1}{2} - \frac{3}{2} = -6$, $\mathfrak{B} = \frac{1}{6}$, $\therefore$ since also $\mathfrak{a} = 2$, $\mathfrak{b} = 3$, $\mathfrak{c} = 2$, consequently $\nu = 7$,

$$`[1^2 2^3 5^2] = \frac{1.2.3.4.5.6}{1.2 \times 1.2.3 \times 1.2} . 6 - 6 . 6^2 + \tfrac{1}{6} . 6^3 = 0.$$

SECTION XCVII.

The symmetrical functions of the roots of the equation $x^n - 1 = 0$ are principally of use in eliminating irrational magnitudes from equations. To eliminate irrational magnitudes from an equation, or to make it rational, implies no more than from this equation to derive another, which only contains rational magnitudes, and is such, that the roots of the former equation are also roots of the latter. In order that it may be seen how this is effected, I shall assume that we have the equation of the first degree $x - A = 0$, in which A denotes any irrational

expression, and that we wish to find for x an equation which is free from irrational magnitudes. Since each irrational magnitude in the expression A, has more values than one, consequently the value of x has various significations, and while nothing nearer than this is determined, this value is doubtful. An equation free from irrational magnitudes which has the expression A for a root, can, on account of this very doubtful signification, neither give this nor that value which we suppose it to have, but must, at the same time, necessarily give all those different values, which this equation contains, because otherwise there would be no sufficient reason why it should give exactly this one, and not every other value likewise. Now, since the values of x are functions of the coefficients of the equation, but the coefficients, according to the condition, are rational, consequently the difference of these values cannot arise from the coefficients, but must be founded in the degree of the equation. The degree of the equation must $\therefore$ be equal to the number of the different values, which the expression A can have. But if the condition respecting the rationality of the coefficients be omitted, then certainly equations of lower degrees may be found, which have this expression for roots, because then the coefficients themselves are undetermined.

For instance, let $x=\sqrt{k}$. The irrational magnitude $\sqrt{k}$ has two roots, viz. $+\sqrt{k}$ and $-\sqrt{k}$. The required equation must $\therefore$ have these two values for roots, and it consequently is $(x+\sqrt{k})\ (x-\sqrt{k})=0$, or, by actual multiplication, $x^2-k=0$; a rational equation, which we

should also have obtained, if we had squared both parts of the equation $x=\sqrt{k}$.

Further, let $x=\sqrt[3]{k}$. If we denote the three roots of the equation $x^3-1=0$ by α, β, γ, then the irrational magnitude $\sqrt[3]{k}$ has the three values $\alpha\sqrt[3]{k}$, $\beta\sqrt[3]{k}$, $\gamma\sqrt[3]{k}$, and these consequently must be the roots of the required equation. It is $\therefore$

$$(x-\alpha\sqrt[3]{k})\,(x-\beta\sqrt[3]{k})\,(x-\gamma\sqrt[3]{k}) = 0.$$

By actual multiplication, we obtain

$$x^3-{}'[1]\sqrt[3]{k}\,.\,x^2+{}'[1^2]\sqrt[3]{k^2}\,.\,x-{}'[1^3]\,k=0\,;$$

or, since by the preceding §, $'[1]=0$, $'[1^2]=0$, $'[1^3]=\frac{1\,.\,2}{1\,.\,2\,.\,3}\,.\,3=1$,

$$x^3-k=0\,;$$

a rational equation, which we should also have found, by raising both parts of the equation $x=\sqrt[3]{k}$ to the third power.

I shall now put $x=\sqrt[4]{k}$. Since $+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, are the four roots of the equation $x^4-1=0$, then $+\sqrt[4]{k}$, $-\sqrt[4]{k}$, $+\sqrt{-1}\,.\,\sqrt[4]{k}$, $-\sqrt{-1}\,.\,\sqrt[4]{k}$, are the four values of $\sqrt[4]{k}$, and the required equation is $\therefore$

$$(x-\sqrt[4]{k})(x+\sqrt[4]{k})(x-\sqrt{-1}.\sqrt[4]{k})(x+\sqrt{-1}.\sqrt[4]{k})=0$$

or by actually performing the multiplication

$$x^4-k=0,$$

as was required.

If for $\sqrt[4]{k}$ we had only taken the two values $+\sqrt[4]{k}$,

$-\sqrt[4]{k}$, and from these had formed the equation $(x-\sqrt[4]{k})(x+\sqrt[4]{k}=0)$ we might have foreseen, at once, that no rational equation could be found. And this is actually the case; thus we obtain $x^2-.\sqrt{k}=0$. Each of the two values of x contains the irrational magnitude $\sqrt{k}$, and since this has a two-fold value, we consequently obtain the four values of x.

Hence we perceive, at least, how we are to proceed with equations of the first degree, in order to render them rational. Thus, if $x=A$ be an equation of this kind, and we denote by A', A'', A''', &c. the different values which the irrational expression A contains, by reason of the many significations which its irrational magnitudes have, then

$$(x-A')(x-A'')(x-A''')\ldots\ldots\ldots=0$$

is always the equation free from irrational magnitudes, which was sought.

The following problems will throw more light on this subject.

SECTION XCVIII.

Prob. Make the equation $x=\sqrt{p}+\sqrt{q}$ rational.

Solution. The irrational expression $\sqrt{p}+\sqrt{q}$ may here have four different values, according as we give the roots $\sqrt{p}$, $\sqrt{q}$, the sign $+$ or $-$, and these values are

$$+\sqrt{p}+\sqrt{q},\ -\sqrt{p}-\sqrt{q},\ +\sqrt{p}-\sqrt{q},\ -\sqrt{p}+\sqrt{q}.$$

The equation free from irrational magnitudes is consequently

$$(x - \sqrt{p} - \sqrt{q})\ (x + \sqrt{p} + \sqrt{q})$$
$$(x - \sqrt{p} + \sqrt{q})\ (x + \sqrt{p} - \sqrt{q}) = 0,$$

or, when we multiply the first and second factors together, as also the third and fourth

$$(x^2 - p - q - 2\sqrt{pq})\ (x^2 - p - q + 2\sqrt{pq}) = 0,$$

or lastly, by completing the multiplication the second time

$$x^4 - 2(p+q)\,x^2 + (p-q)^2 = 0$$

SECTION XCIX.

Prob. Make the equation $x = a\sqrt{p} + \frac{b}{\sqrt{p}}$ rational.

Solution. Since in this equation there is only one irrational magnitude, and that the square root, viz. $\sqrt{p}$, consequently x can have no more than two values, and these are

$$a\sqrt{p} + \frac{b}{\sqrt{p}},\quad -a\sqrt{p} - \frac{b}{\sqrt{p}}.$$

The rational equation is $\therefore$

$$\left(x - a\sqrt{p} - \frac{b}{\sqrt{p}}\right)\ \left(x + a\sqrt{p} + \frac{b}{\sqrt{p}}\right) = 0$$

or $$x^2 - a^2p - 2ab - \frac{b^2}{p} = 0$$

or likewise $px^2 - (ap+b)^2 = 0$

SECTION C.

Prob. Make the equation $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$ rational.

Solution 1. The cubic irrational magnitude $\sqrt[3]{p}$, may have three values viz.

$$a\sqrt[3]{p},\ \beta\sqrt[3]{p},\ \gamma\sqrt[3]{p},$$

when a, β, γ denote the three roots of the equation $x^3-1=0$, unity included. These three values of $\sqrt[3]{p}$ correspond to the three following values of its square $\sqrt[3]{p^2}$;

$$a^2\sqrt[3]{p^2},\ \beta^2\sqrt[3]{p^2},\ \gamma^2\sqrt[3]{p^2}.$$

Consequently x can contain no more than these three values:

$$aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2},\ \beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2},\ \gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}.$$

The rational equation is $\therefore$ of the third degree.

2. It is represented by

$$x^3 - Px^2 + Qx - R = 0$$

then

$$\begin{aligned}
P &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})+(\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2}) \\
&\qquad + (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}) \\
&= {}'[1]a\sqrt[3]{p}+{}'[2]b\sqrt[3]{p^2} \\
Q &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2})+ \\
&\quad (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2})+ \\
&\quad (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2})\ (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}) \\
&= {}'[1^2]a^2\sqrt[3]{p^2}+{}'[12]abp+{}'[2^2]\,.\,b^2p\sqrt[3]{p} \\
R &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2}) \\
&\qquad (\gamma a\sqrt[3]{p}+\gamma^2\sqrt[3]{p^2}) \\
&= {}'[1^3]a^3p+{}'[1^22]a^2bp\sqrt[3]{p}+{}'[12^2]ab^2p\sqrt[3]{p^2} \\
&\qquad + {}'[2^3]\,b^3p^2
\end{aligned}$$

3. Here, then, the numerical expressions $`[1]$, $`[2]$, $`[1^2]$, $`[2^2]$, $`[1^2 2]$, $`[12^2]$, $`[12]$, $`[1^3]$, $`[2^3]$ occur, the first six of which vanish, by 2, § XCV. Further, by the same §, since here $n=3$, $`[12]=-3$, $`[1^3]=1$, $`[2^3]=1$. By the substitution of these values in the expressions for P, Q, R, we obtain $P=0$, $Q=-3abp$, $R=a^3p+b^3p^2$, and hence the required rational equation

$$x^3 - 3abpx - a^3p - b^3p^2 = 0.$$

Remark. We could essentially have shortened the calculation for determining the values of P, Q, R, by omitting at once all those terms in which p is included under the radical sign, because it might have been foreseen that they vanish in the results, as the required equation must contain no irrational magnitudes.

SECTION CI.

The problem in the preceding § leads immediately to the solution of equations of the third degree. For since the supposed roots $\alpha a\sqrt[3]{p}+\alpha^2 b\sqrt[3]{p^2}$, $\beta a\sqrt[3]{p}+\beta^2 b\sqrt[3]{p^2}$, $\gamma a\sqrt[3]{p}+\gamma^2 b\sqrt[3]{p^2}$ led to the equation $x^3-3abpx-a^3p-b^3p^2=0$, it may be inferred conversely, that every equation of this form must have these three roots. If we put $p=1$, then this equation is transformed into

$$x^3 - 3abx - a^3 - b^3 = 0$$

and the three roots of this equation are consequently

$$\alpha a + \alpha^2 b,\ \beta a + \beta^2 b,\ \gamma a + \gamma^2 b.$$

Since one of the three roots α, β, γ must be equal to

unity, we may $\therefore$ put $\gamma=1$; farther, if $\alpha^2=\beta$, $\beta^2=\alpha$; the three roots of the equation $x^3-3abx-a^3-b^3=0$ consequently assume the following form:

$$\alpha a + \beta b,\ \ \beta a + \alpha b,\ \ a + b.$$

We obtain also precisely the same result from Cardan's Formula, which, as is already known, tries to reduce the given equation to the form $x^3-3abx-a^3-b^3=0$.

SECTION CII.

The problem, § C, may also be solved by another method. Thus put $\sqrt[3]{p}=y$, then $\sqrt[3]{p^2}=y^2$, $\therefore$ $x=ay+by^2$: the value of y determines that of x. But y has three values, viz. $\alpha\sqrt[3]{p}$, $\beta\sqrt[3]{p}$, $\gamma\sqrt[3]{p}$, which are all comprehended in the equation $y^3-p=0$; consequently the values of x must be the result of the elimination of y in the two equations

$$\text{I.}\ \ y^3-p=0$$
$$\text{II.}\ \ x-ay-by^2=0$$

In order to perform this elimination, we give, by § LXXVI, to the equation II. the form $1+(1)y+(2)y^2=0$, so that $(1)=-\frac{a}{x}$, $(2)=-\frac{b}{x}$, and we then obtain the following equation:

$$0=1+(1)[1]+(2)[2]+(12)[12]+(2^2)[2^2]$$
$$+(1^2)[1^2]+(1^3)[1^3]+(1^22)[1^22]$$
$$+(12^2)[12^2]+(2^3)[2^3]$$

The numerical expressions may be taken from the annexed tables, if we put $A=0$, $B=0$, $C=p$. If after this we

substitute again for (1), (2), their values $-\frac{a}{x}$, $-\frac{b}{x}$, we then obtain the equation $x^3-3abpx-b^3p-a^3p^2=0$, as in § C.

SECTION CIII.

PROB. Make the equation $x=a\sqrt[3]{p}+b\sqrt[3]{q}$ rational.

Solution 1. The cubic irrational magnitude $\sqrt[3]{p}$ has three values, viz. $\alpha\sqrt[3]{p}$, $\beta\sqrt[3]{p}$, $\gamma\sqrt[3]{p}$. In like manner $\sqrt[3]{q}$ contains the values $\alpha\sqrt[3]{q}$, $\beta\sqrt[3]{q}$, $\gamma\sqrt[3]{q}$. Each of the first may be combined with each of the latter, and this gives nine values of x, viz.

$$\begin{array}{lll}
\alpha a\sqrt[3]{p}+\alpha b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\alpha b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\alpha b\sqrt[3]{q} \\
\alpha a\sqrt[3]{p}+\beta b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\beta b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\beta b\sqrt[3]{q} \\
\alpha a\sqrt[3]{p}+\gamma b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\gamma b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\gamma b\sqrt[3]{q}.
\end{array}$$

Hence we can find the rational equation in the usual way. But we can also attain this object by elimination as in the preceding §.

2. With this view, put $a\sqrt[3]{p}=y$, $b\sqrt[3]{q}=z$; then $x=y+z$. Now, since y has the values $\alpha a\sqrt[3]{p}$, $\beta a\sqrt[3]{p}$, $\gamma a\sqrt[3]{p}$, and z the values $\alpha b\sqrt[3]{q}$, $\beta b\sqrt[3]{q}$, $\gamma b\sqrt[3]{q}$, which are all included in the two equations $y^3-a^3p=0$, $z^3-b^3q=0$, it merely amounts to this, from the three equations

$$\text{I.} \quad x = y + z$$
$$\text{II.} \quad y^3 - a^3p = 0$$
$$\text{III.} \quad z^3 - b^3q = 0$$

to eliminate the magnitudes y and z.

3. Raise the equation I. to the third power, and put for y^3, z^3, their values a^3p, b^3q from II. and III., also x for $y+z$, we then obtain

$$x^3 = a^3p + b^3q + 3yzx$$
$$\text{or } x^3 - a^3p - b^3q = 3yzx$$

4. Raise this equation again to the third power, then we get

$$(x^3 - a^3p - b^3q)^3 = 27y^3z^3x^3$$

and if for y^3, z^3 we substitute their values

$$(x^3 - a^3p - b^3q)^3 = 27a^3b^3pqx^3.$$

5. If this equation be solved and properly arranged, we obtain

$$x^9 - 3(a^3p + b^3q)x^6 + [3(a^3p + b^3q)^2 - 27a^3b^3pq]\,x^3 - (a^3p + b^3q)^3 = 0,$$

and since this is of the ninth degree, it consequently is, as appears from 1, the most simple rational equation which can be deduced from $x = a\sqrt[3]{p} + b\sqrt[3]{q}$.

SECTION CIV.

PROB. Make the equation $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$ rational.

Solution 1. Since the quadratic irrational magnitudes

may be assumed either positive or negative, their combination gives for x the following eight roots of the required equation:

$$\begin{array}{ll} \sqrt{p} + \sqrt{q} + \sqrt{r}, & -\sqrt{p} - \sqrt{q} - \sqrt{r} \\ \sqrt{p} + \sqrt{q} - \sqrt{r}, & -\sqrt{p} - \sqrt{q} + \sqrt{r} \\ \sqrt{p} - \sqrt{q} + \sqrt{r}, & -\sqrt{p} + \sqrt{q} - \sqrt{r} \\ \sqrt{p} - \sqrt{q} - \sqrt{r}, & -\sqrt{p} + \sqrt{q} + \sqrt{r} \end{array}$$

2. Since here every two roots which are opposite each other only differ in these signs, the equation can only contain even powers of x, and it has $\therefore$, when we put $x^2 = y$, the following form:

$$y^4 - Ay^3 + By^2 - Cy + D = 0$$

and the roots of this equation are

$$\begin{array}{ll} (\sqrt{p} + \sqrt{q} + \sqrt{r})^2, & (\sqrt{p} + \sqrt{q} - \sqrt{r})^2 \\ (\sqrt{p} - \sqrt{q} + \sqrt{r})^2, & (\sqrt{p} - \sqrt{q} - \sqrt{r})^2 \end{array}$$

3. In order to determine from hence the coefficients A, B, C, D, we only need take the sum of these roots, the sum of every two of them, and so on. The following treatment, which has been frequently made use of already in the preceding part of this work, leads to the object in a shorter way. Let $S1$, $S2$, $S3$, $S4$, denote the sum of these roots, the sum of their squares, cubes, and fourth powers; then, when in § IX, $-A$ and $-C$ are put for A and C, and the symbol S for the one [ ] there used,

$$A = S1$$

$$B = \frac{AS1 - S2}{2}$$

$$C = \frac{BS1 - AS2 + S3}{3}$$

$$D = \frac{CS1 - BS2 + AS3 - S4}{4}$$

4. The expressions $S1$, $S2$, $S3$, $S4$, must necessarily be rational, because otherwise the coefficients A, B, C, D, could not be rational. Consequently the irrational magnitudes in the solution must alternately be left out, and they may ∴ be entirely omitted in the calculation. With reference to this remark, and since we treat the trinomial $\sqrt{p} + \sqrt{q} + \sqrt{r}$ as a binomial $\sqrt{p} + (\sqrt{q} + \sqrt{r})$, the calculation stands thus:

$$(\sqrt{p}+\sqrt{q}+\sqrt{r})^2 = p+q+r+\&c.$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^4 &= p^2+6p(\sqrt{q}+\sqrt{r})^2+(\sqrt{q}+\sqrt{r})^4\ \&c.\\ &= p^2+6p(q+r)+q^2+6qr+r^2+\&c.\\ &= p^2+q^2+r^2+6(pq+pr+qr)+\&c.\end{aligned}$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^6 &= p^3+15p^2(\sqrt{q}+\sqrt{r})^2+15p(\sqrt{q}+\sqrt{r})^4+(\sqrt{q}+\sqrt{r})^6+\&c.\\ &= p^3+15p^2(q+r)+15p(q^2+6qr+r^2)+q^3+15q^2r+15qr^2+r^3+\&c.\\ &= p^3+q^3+r^3+15(p^2q+pq^2+p^2r+pr^2+q^2r+qr^2)+99pqr+\&c.\end{aligned}$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^3 &= p^4+28p^3(\sqrt{q}+\sqrt{r})^2+70p^2(\sqrt{q}+\sqrt{r})^4+28p(\sqrt{q}+\sqrt{r})^6+(\sqrt{q}+\sqrt{r})^8+\&c.\\ &= p^4+28p^3(q+r)+70p^2(q^2+6qr+r^2)+28p(q^3+15q^2r+15qr^2+r^3)\\ &\quad +q^4+28q^3r+70q^2r^2+28qr^3+r^4\ \&c.\\ &= p^4+q^4+r^4+28(p^3q+pq^3+p^3r+pr^3+q^3r+qr^3)+70(p^2q^2+p^2r^2+q^2r^2)\\ &\quad +420(pqr^2+pq^2r+p^2qr)+\&c.\end{aligned}$$

5. It is easily seen, that if, instead of $\sqrt{p}+\sqrt{q}+\sqrt{r}$, we had raised every alternate one of the expressions $\sqrt{p}+\sqrt{q}-\sqrt{r}$, $\sqrt{p}-\sqrt{q}+\sqrt{r}$, $\sqrt{p}-\sqrt{q}-\sqrt{r}$ to the same powers, the rational parts would have been the same. Now, since in $S1$, $S2$, $S3$, $S4$, the rational terms must be left out, we obtain

$S1=4\ (p+q+r)$
$S2=4\ [p^2+q^2+r^2+6(pq+pr+qr)]$
$S3=4\ [p^3+q^3+r^3+15(p^2q+pq^2+p^2r+pr^2+q^2r+qr^2)+90pqr]$
$S4=4\ [p^4+q^4+r^4+28\ (p^3q+pq^3+p^3r+pr^3+q^3r+qr^3)+70\ (p^2q^2+p^2r^2+q^2r^2)+420\ (pqr^2+pq^2r+p^2qr)]$

or more briefly, when the brackets [ ] refer to the magnitudes p, q, r,

$S1=4\ [1]$
$S2=4\ ([2]+6\ [1^2])$
$S3=4\ ([3]+15\ [12]+90\ [1^3])$
$S4=4\ ([4]+28\ [13]+70\ [2^2]+420\ [1^22])$

6. If these values be substituted in the equations in 3, we then obtain the coefficients A, B, C, D, expressed by the given magnitudes p, q, r.

SECTION CV.

PROB. Make the equation $x=\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}$ rational.

Solution 1. It may be shown by inferences, as is in 1 and 2 of the preceding §, that, when we put $x^2=y$, the rational equation is of the form

$$y^8 - Ay^7 + By^6 - Cy^5 + Dy^4 - Ey^3 + Fy^2 - Gy + H = 0$$

and has the following expressions for roots:

$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} - \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} - \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} + \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} + \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} - \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} - \sqrt{r} - \sqrt{s})^2$$

2. Give the symbol S the meaning which it had in the preceding §, and first of all try to find the expressions $S1$, $S2$, $S3$, ... $S8$. Since the calculation is managed in the same way as in the preceding §, I shall not detain my readers with it, but only remind them, that in the involution the expression $\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s}$ may be considered as a binomial, whose two parts are $\sqrt{s}$, and $\sqrt{p} + \sqrt{q} + \sqrt{r}$.

3. Consequently we have

$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^2 =$$
$$s + (\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + \&c.$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^4 =$$
$$s^2 + 6s(\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + (\sqrt{p} + \sqrt{q} + \sqrt{r})^4 + \&c.$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^6 =$$
$$s^3 + 15s^2(\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + 15s(\sqrt{p} + \sqrt{q} + \sqrt{r})^4 + (\sqrt{p} + \sqrt{q} + \sqrt{r})^6 + \&c.$$

&c.

or, when the developement of the powers of $\sqrt{p}+\sqrt{q}+\sqrt{r}$ in 4 of the preceding §, are used,

$$(\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s})^2 =$$
$$p+q+r+s+\&c.$$
$$(\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s})^4 =$$
$$p^2+q^2+r^2+s^2+6(pq+pr+ps+qr+qs+rs)+\&c.$$
$$(\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s})^6 =$$
$$p^3+q^3+r^3+s^3+15\,(p^2q+pq^2+p^2r+pr^2+p^2s+ps^2$$
$$+q^2r+qr^2+q^2s+qs^2+r^2s+rs^2)\;+90\,(pqr+pqs+$$
$$prs+qrs+\&c.$$
$$\&c.$$

in which only the irrational terms have been omitted.

4. For the same reasons as in 5 of the preceding §, we obtain from hence

$$S1 = 8\,[1]$$
$$S2 = 8\,([2]\;+\;6\,[1^2])$$
$$S3 = 8\,([3]\;+\;15\,[12]\;+\;90\,[1^3])$$
$$\&c.$$

and the substitution of these values in the formulæ in 3 of the preceding §, which must be extended for this purpose, gives the coefficients A, B, C, &c.

SECTION CVI.

Prob. Make the following equation of the first degree, with an indeterminate number of quadratic irrational magnitudes, rational, viz.

$$x = \sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}+\ldots\ldots+\sqrt{w}.$$

Solution 1. By the two preceding §§ it is easily

inferred, that when n is the number of the irrational magnitudes $\sqrt{p}$, $\sqrt{q}$, ... $\sqrt{w}$, the degree of the rational equation is equal to the power 2^n. But since the different values of x are such, that two of them are always similar, but with different signs, the equation consequently is only of the 2^{n-1}th degree, when we put $x^2 = y$.

2. The conclusions in the two preceding §§, when extended, give the following results:

$$S1 = 2^{n-1}[1]$$
$$S2 = 2^{n-1}([2] + 6[1^2])$$
$$S3 = 2^{n-1}([3] + 15[12] + 90[1^3])$$
$$S4 = 2^{n-1}([4] + 28[13] + 70[2^2] + 420[1^2 2] + 2520[1^4])$$
$$S5 = 2^{n-1}([5] + 45[14] + 210[23] + 1260[1^2 3] + 3150[12^2] + 18900[1^3 2] + 113400[1^3])$$

&c.

3. Hence the law of the formation is easily perceived. As an example, I will take $S5$. The number 5, and its divisions into combinations of two, three, &c, give the numerical expressions $[5]$, $[14]$, $[23]$, $[1^2 3]$, $[12^2]$, $[1^3 2]$, $[1^5]$. The coefficients are no other than the number of transpositions of different things, whose repeating exponents are twice as great as the radical exponents of the numerical expression; consequently the coefficients of $[5]$, $[14]$, $[23]$, $[1^2 3]$, $[12^2]$, $[1^3 2]$, $[1^5]$, the number of transpositions of the different things a^{10}, a^2b^8, a^4b^6, $a^2b^2c^6$, $a^2b^4c^4$, $a^2b^2c^2d^4$, $a^2b^2c^2d^2e^2$, or 1, 45, 210, 1260, 3150, 18900, 113400. Those of my readers, who understand the polynomial theorem, will not have the

least difficulty in comprehending the reason of this. For by the two preceding sections, the expressions $S1$, $S2$, $S3$, &c., when 2^{n-r} is left out, are no other than the developments of the second, fourth, sixth, eighth, &c. powers of $\surd p + \surd q + \surd r + \ldots\ldots + \surd w$, with the omission of all those terms which contain irrational magnitudes, or, which is the same, the developments of the even powers of $p+q+r+\ldots\ldots+w$ with the omission of all those terms in which there are odd exponents, and by dividing the exponents in the remaining ones by two.

4. Now, if we put $2^n = m$, then the required rational equation

$$x^m - Ax^{m-2} + Bx^{m-4} - Cx^{m-6} + \&c. = 0$$

and the coefficients A, B, C, &c. are determined by the following equations:

$$\begin{aligned} A &= S1 \\ 2B &= AS1 - S2 \\ 3C &= BS1 - AS2 + S3 \\ &\&c. \end{aligned}$$

Remark. To this belongs the celebrated problem which Fermat proposed to the analysts of his time, and to the solution of which he more particularly challenged Descartes. It is this from the equation

$$ab = \surd(ab - a^2) + \surd(a^2 + ad + s^2) + \surd ma + \surd(d^2 - a^2) - \surd(ar + a^2)$$

to take away the irrational magnitudes. It is only necessary to substitute x for ab, and for the compound magnitudes under the radical signs to put the monomials p, q, r, s, t; then it only remains to make the equation $x = \surd p + \surd q + \surd r + \surd s + \surd t$ rational, and in the

equation thus obtained, for x to substitute again its values p, q, r, s, t.

SECTION CVII.

Prob. Make the equation $x=a\sqrt[4]{p}+b\sqrt[4]{p^2}+c\sqrt[4]{p^3}$ rational.

Solution 1. Put $\sqrt[4]{p}=y$, then this equation is

$$x - ay - by^2 - cy^3 = 0.$$

Now, since y contains four values, viz. $+y$, $-y$, $+y\sqrt{-1}$, $-y\sqrt{-1}$, we get the four following equations, all of which obtain at the same time.

$$\begin{aligned}
&x - ay - by^2 - cy^3 = 0\\
&x + ay - by^2 + cy^3 = 0\\
&x - ay\sqrt{-1} + by^2 + cy^3\sqrt{-1} = 0\\
&x + ay\sqrt{-1} + by^2 - cy^3\sqrt{-1} = 0
\end{aligned}$$

or

$$\begin{aligned}
&(x - by^2) - (ay + cy^3) = 0\\
&(x - by^2) + (ay + cy^3) = 0\\
&(x + by^2) - (ay - cy^3)\sqrt{-1} = 0\\
&(x + by^2) + (ay - cy^3)\sqrt{-1} = 0
\end{aligned}$$

The equation sought must $\therefore$ be the product of these.

2. If the two first and the two last be multiplied together, we obtain

$$\begin{aligned}
&x^2 - 2by^2x + b^2y^4 - a^2y^2 - 2acy^4 - c^2y^6 = 0\\
&x^2 + 2by^2x + b^2y^4 + a^2y^2 - 2acy^4 + c^2y^6 = 0
\end{aligned}$$

or if in these equations we substitute p for y^4,

$$[x^2 + (b^2 - 2ac)p] - (2bx + a^2 + c^2p)y^2 = 0$$
$$[x^2 + (b^2 - 2ac)p] + (2bx + a^2 + c^2p)y^2 = 0$$

3. If we multiply these equations, and then put p for y^4, we obtain the required rational equation of the fourth degree,

$$x^4 - 2(b^2 + 2ac)px^2 - 4(a^2 + c^2p)bpx$$
$$+ (b^2 - 2ac)^2p^2 - (a^2 + c^2p)^2p = 0$$

and the four roots of this equation are

$$\begin{array}{l} a\sqrt[4]{p} \qquad + b\sqrt[4]{p^2} + c\sqrt[4]{p^3} \\ -a\sqrt[4]{p} \qquad + b\sqrt[4]{p^2} - c\sqrt[4]{p^3} \\ a\sqrt[4]{p}\,.\,\sqrt{-1} - b\sqrt[4]{p^2} - c\sqrt[4]{p^3}\,.\,\sqrt{-1} \\ -a\sqrt[4]{p}\,.\,\sqrt{-1} - b\sqrt[4]{p^2} + c\sqrt[4]{p^3}\,.\,\sqrt{-1} \end{array}$$

Corollary When ∴ an equation of the fourth degree has the form just found, then its four roots may always be determined without any further calculation. I shall now show, that, presupposing the solution of cubic equations to be known, every equation of the fourth degree can have this form.

SECTION CVIII.

Let

$$x^4 - Ax^2 - Bx - C = 0$$

be the equation to be solved: it is general, because in every equation the second term, if there be such, may be omitted. If this equation be identical with that in 3

of the preceding §, we then must have

$$\text{I.}\quad 2p(b^2+2ac) = A$$
$$\text{II.}\quad 4bp(a^2+c^2p) = B$$
$$\text{III.}\quad (a^2+c^2p)^2p \ldots (b^2-2ac)^2p^2 = C$$

The two first equations give

$$(b^2+2ac)p = \frac{A}{2}$$

$$a^2 + c2p = \frac{B}{4bp}$$

and if we make use of these values in the equation III, after having previously given it the form

$$(a^2+c^2p)^2p - (b^2+2ac)^2p^2 + 8ab^2cp^2 = C$$

we then obtain

$$\frac{B^2}{16b^2p} - \frac{A^2}{4} + 8ab^2cp^2 = C$$

From the equation I we also obtain

$$\text{IV.}\quad 4acp = A - 2b^2p$$

and the substitution of this value in the equation just found, gives

$$\frac{B^2}{16b^2p} - \frac{A^2}{4} + 2Ab^2p - 4b^4p^2 = C$$

Since the three equations I, II, III, contain four indeterminate magnitudes a, b, c, p, we can ∴ assume any one of them. Put $b = 1$, then, after getting rid of the denominator,

$$B^2 - 4A^2p + 32Ap^2 - 64p^3 = 16Cp$$

or, when arranged according to p,

$$V.\; p^3 - \tfrac{1}{2}Ap^2 + \tfrac{1}{4}(C + \tfrac{1}{4}A^2)p - \tfrac{1}{64}B^2 = 0$$

an equation of the third degree, which merely contains the unknown magnitude p.

From the equations II and IV we obtain, when we put $b = 1$, and divide the latter by $2\sqrt{p}$,

$$a^2 + c^2 p = \frac{B}{4p},\ 2ac\sqrt{p} = \frac{A-2p}{2\sqrt{p}}$$

and when we add the second to the first, and also subtract the one from the other, then again extract the square root from the sum and the remainder

$$a + c\sqrt{p} = \sqrt{\left(\frac{B}{4p} + \frac{A}{2\sqrt{p}} - \sqrt{p}\right)}$$

$$a - c\sqrt{p} = \sqrt{\left(\frac{B}{4p} - \frac{A}{2\sqrt{p}} + \sqrt{p}\right)}$$

But the four roots in 3 of the preceding §, when we put $b=1$, has the following form

$$\sqrt{p} + (a + c\sqrt{p})\sqrt[4]{p}$$

$$\sqrt{p} - (a + c\sqrt{p})\sqrt[4]{p}$$

$$-\sqrt{p} + (a - c\sqrt{p})\sqrt[4]{p}\,.\sqrt{-1}$$

$$-\sqrt{p} - (a - c\sqrt{p})\sqrt[4]{p}\,.\sqrt{-1}$$

and when in this we substitute for $a + c\sqrt{p}$, $a - c\sqrt{p}$ their values, we obtain the following roots of the equation $x^4 - Ax^2 - Bx - C = 0$:

$$\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(B\sqrt{p} + 2Ap - 4p^2)}$$

$$\sqrt{p} - \frac{1}{2\sqrt{p}}\sqrt{(B\sqrt{p} + 2Ap - 4p^2)}$$

$$-\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(-\sqrt{p} + 2Ap - 4p^2)}$$

$$-\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(-\sqrt{p} + 2Ap - 4p^2)}$$

Having $\therefore$ already determined the value of p from the equation V, we also obtain the roots of the given equation. Besides, it is exactly the same which of the three values of p we make use of, because in each case we must necessarily always get the same roots.

SECTION CIX.

Prob. Make the equation

$$x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$$

rational

Solution 1. When we put $\sqrt[5]{p} = y, \therefore y^5 - p = 0$, then the equation is transformed into

$$x - ay - by^2 - cy^3 - dy^4 = 0.$$

Now y has five values, viz. αy, βy, γy, δy, εy, when α, β, γ, δ, ε, denote the five roots of the equation $y^5 - 1 = 0$ (unity included); we have $\therefore$ the five following distinct equations:

$$x - \alpha ay - \alpha^2 by^2 - \alpha^3 cy^3 - \alpha^4 dy^4 = 0$$
$$x - \beta ay - \beta^2 by^2 - \beta^3 cy^3 - \beta^4 dy^4 = 0$$
$$x - \gamma ay - \gamma^2 by^2 - \gamma^3 cy^3 - \gamma^4 dy^4 = 0$$
$$x - \delta ay - \delta^2 by^2 - \delta^3 cy^3 - \delta^4 dy^4 = 0$$
$$x - \varepsilon ay - \varepsilon^2 by^2 - \varepsilon^3 cy^3 - \varepsilon^4 dy^4 = 0$$

and their product will give the required rational equation, if we again put $\sqrt[5]{p}$ for y.

2. In fact, this implies no more than to eliminate x from the two equations $x - ay - by^2 - cy^3 - dy^4 = 0$, $y^5 - p = 0$, and consequently in this case all the methods of elimination in the preceding chapter are applicable.

If we make use of Cramer's method as the easiest, we shall arrive at numerical expressions, which exceed the limits of the annexed tables, and $\therefore$ must be calculated. But we arrive at this object in a much shorter way by managing the calculation in such a way, that the numerical expressions refer only to the roots of unity, because these are more easily calculated.

3. For this purpose we only require from the two equations

$$\text{I. } z^5 - 1 = 0$$

$$\text{II. } x - ayz - by^2z^2 - cy^3z^3 - dy^4z^4 = 0$$

to eliminate the magnitude z; for if we substitute in II the five values α, β, γ, δ, ε, of z from the first, we then obtain the same equations as in 1.

4. In order to be able to apply Cramer's method of elimination (§ LXXVI), we give the equation II the form

$$1 + (1)\,z + (2)\,z^2 + (3)\,z^3 + (4)\,z^4 = 0$$

then $(1) = -\frac{ay}{x}$, $(2) = -\frac{by^2}{x}$, $(3) = -\frac{cy^3}{x}$, $(4) = -\frac{dy^4}{x}$,

5. Since the numerical expressions in the equation, § LVIII, LIX, (c), in the present case relate to the roots of the equation $z^5 - 1 = 0$, then all those in which the sum of the radical exponents is not divisible by five vanish, by 2, § XCV. With reference to this remark, we obtain the following final equation:

$$
\begin{aligned}
o = 1 &+ (14)'[14] + (24^2)'[24^2] + (34^3)'[34^3] + (4^5)'[4^5] \\
&+ (23)'[23] + (3^24)'[3^24] + (124^3)'[124^3] \\
&+ (1^23)'[1^23] + (1^24^2)'[1^24^2] + (13^24^2)'[13^24^2] \\
&+ (12^2)'[12^2] + (1234)'[1234] + (2^234^2)'[2^234^2] \\
&+ (1^32)'[1^32] + (13^3)'[13^3] + (23^34)'[23^34] \\
&+ (1^5)'[1^5] + (2^34)'[2^34] + (3^5)'[3^5] \\
& + (2^23^2)'[2^23^2] \\
& + (1^334)'[1^334] \\
& + (1^22^24)'[1^22^24] \\
& + (1^223^2)'[1^223^2] \\
& + (12^33)'[12^33] \\
& + (2^5)'[2^5] .
\end{aligned}
$$

6. The numerical expressions in this equation may also be calculated by § XCV, which, indeed, is not difficult for the present case. If after this again, we put for the symbols (1), (2), (3), (4), their values from 4, likewise p for y^5, and multiply the equation by x^5, we then obtain

$$
\begin{aligned}
o = {} & x^5 \left.\begin{array}{l} -5ad \\ -5bc \end{array}\right) px^3 \left.\begin{array}{l} -5bd^2 \\ -5c^2d \end{array}\right) p^2x - 5cd^3p^3x - d^5p^4 \\
& \left.\begin{array}{l} -5a^2c \\ -5ab^2 \end{array}\right) px^2 \left.\begin{array}{l} -5a^2d^2 \\ -5abcd \\ -5ac^3 \\ -5b^3d \\ +5b^2c^2 \end{array}\right\} p^2x \left.\begin{array}{l} +5abd^3 \\ -5ac^2d^2 \\ -5b^2cd^2 \\ +5bc^3d \\ -c^5 \end{array}\right\} p^3 \\
& -5a^3bpx \\
& -a^5p \\
& \left.\begin{array}{l} +5a^3cd \\ -5a^2b^2d \\ -5a^2bc^2 \\ +5ab^3c \\ -b^5 \end{array}\right\} p^2
\end{aligned}
$$

7. If ∴ the required rational equation be represented by

$$x^5 - Ax^3 - Bx^2 - Cx - D = 0$$

then

$$A = 5\,(ad + bc)\,p$$

$$B = 5\,(a^2c + ab^2 + bd^2p + c^2dp)\,p$$

$$C = 5\,(a^3b + b^3dp + ac^3p + cd^3p^2)\,p - 5\,(a^2d^2 + b^2c^2)\,p^2 + 5\,abcdp^2$$

$$D = a^5p + b^5p^2 + c^5p^3 + d^5p^4 - 5\,(a^3cd + ab^3c + bc^3dp + abd^3p)\,p^2 + 5\,(a^2b^2d + a^2bc^2 + ac^2d^2p + b^2cd^2p)\,p^2$$

SECTION CX.

The equation $x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$ led to an equation of the fifth degree, of the form in 7 of the preceding §; and the five roots of this last equation are consequently

$$x = \alpha a\sqrt[5]{p} + \alpha^2 b\sqrt[5]{p^2} + \alpha^3 c\sqrt[5]{p^3} + \alpha^4 d\sqrt[5]{p^4}$$

$$x = \beta a\sqrt[5]{p} + \beta^2 b\sqrt[5]{p^2} + \beta^3 c\sqrt[5]{p^3} + \beta^4 d\sqrt[5]{p^4}$$

$$x = \gamma a\sqrt[5]{p} + \gamma^2 b\sqrt[5]{p^2} + \gamma^3 c\sqrt[5]{p^3} + \gamma^4 d\sqrt[5]{p^4}$$

$$x = \delta a\sqrt[5]{p} + \delta^2 b\sqrt[5]{p^2} + \delta^3 c\sqrt[5]{p^3} + \delta^4 d\sqrt[5]{p^4}$$

$$x = \varepsilon a\sqrt[5]{p} + \varepsilon^2 b\sqrt[5]{p^2} + \varepsilon^3 c\sqrt[5]{p^3} + \varepsilon^4 d\sqrt[5]{p^4}$$

Therefore, conversely, if an equation of the fifth degree has the given form, we have its roots in its stead. If ∴ we could reduce every given equation of the fifth degree to this form, we should then have the general solution of equations of this degree. To effect this, it is indispen-

sably necessary, from the given coefficients A, B, C, D, by means of the equations in 7 of the preceding §, to be able to determine the magnitudes a, b, c, d, p, one of which is arbitrary, in a similar way with that in § CVIII in the case of equations of the fourth degree, and also in § C, where the transformed equation had Cardan's form. But all the endeavours of the greatest Analysts to attain this object have been fruitless, and we shall see in the sequel, why it must be the case. However, a treatise by Euler on the general solution of equations, and particularly those of the fifth degree, may always be read with pleasure and instruction; it is to be found in the ninth part of the new Petersburg Commentaries, and also in the third Part of Michelsen's Translation of Euler's Introduction.

SECTION CXI.

Although, however, we cannot obtain the general solution of equations of the fifth degree by the method in the preceding §, yet there are several particular equations, to which this solution is applicable, of which I shall, with Euler, only adduce those which do not lead to very complicated forms.

I. If in the equations in 7, § CIX, we put $c = 0$, $d = 0$, we then have

$$A = 0,\ B = 5ab^2p,\ C = 5a^3bp,$$
$$D = a^5p + b^5p^2.$$

From the second and third of these equations we obtain

$$a^5p = \frac{C^2}{5B},\ b^5p^2 = \frac{B^3}{25C}$$

hence

$$D = \frac{C^2}{5B} + \frac{B^3}{25C}$$

Also

$$a\sqrt[5]{p} = \sqrt[5]{\frac{C^2}{5B}}, \; b\sqrt[5]{p^2} = \sqrt[5]{\frac{B^3}{25C}}$$

If $\therefore$ the equation

$$x^5 - Bx^2 - Cx - \frac{C^2}{5B} - \frac{B^3}{25C} = 0$$

be given, then

$$\alpha\sqrt{\frac{C^2}{5B}} + \alpha^2\sqrt{\frac{B^3}{25C}}$$

is one of its roots, and the remaining roots are obtained by substituting β, γ, δ, ε successively for α.

We should in like manner have found the same equation and the same roots, if we had put a and b, or a and c, or b and $d = 0$. Thus if we put $b = 0$, and $d = 0$, we have

$$A = 0, \; B = 5a^2cp, \; C = 5ac^3p^2$$
$$D = a^5p + c^5p^3.$$

From the second and third equation we obtain

$$a^5p = \frac{B^3}{25C}, \; c^5p^3 = \frac{C^2}{5B}$$

and these give

$$D = \frac{B^3}{25C} + \frac{C^2}{5B}$$

$$a\sqrt[5]{p} = \sqrt[5]{\frac{B^3}{25C}}, \; c\sqrt[5]{p^3} = \sqrt[5]{\frac{C^2}{5B}}$$

We have $\therefore$ again the equation

$$x^5 - Bx^2 - Cx - \frac{B^3}{25C} - \frac{C^2}{5B} = 0$$

and one of its roots

$$= \alpha a\sqrt[5]{p} + \alpha^3 c\sqrt[5]{p^3} = \alpha\sqrt[5]{\frac{B^3}{25C}} + \alpha^3\sqrt[5]{\frac{C^2}{5B}}$$

The remaining ones are obtained by substituting $\beta, \gamma, \delta, \varepsilon$, successively for α. Moreover, that the five roots, which we find by these means, are not different from those already found, we may easily convince ourselves by putting $\alpha^2, \alpha^3, \alpha^4, \alpha^5$ ($= 1$) for $\beta, \gamma, \delta, \varepsilon$ (§ LXXXVII).

II. If in the equations in 7, § CVIII, we put $b = o$, and $c = o$, we obtain

$$A = 5\,adp,\ B = o,\ C = -\ 5a^2d^2p^2,$$
$$D = a^5p + d^5p^4$$

The first and third of these equations give

$$C = -\frac{A^2}{5}$$

Further, the fourth gives

$$D^2 = (a^5p + d^5p^4)^2 = (a^5p - d^5p^4)^2 + 4a^5d^5p^5$$
$$= (a^5p - d^5p^4)^2 + 4\left(\frac{A}{5}\right)^5$$

consequently

$$a^5p - d^5p^4 = \sqrt{\left[D^2 - 4\left(\frac{A}{5}\right)^5\right]}$$

Now since

$$a^5p + d^5p^4 = D$$

then

$$a^5p = \tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}$$

$$d^5p^4 = \tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}$$

and $\therefore$

$$a\sqrt[5]{p} = \sqrt[5]{\left[\tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]}$$

$$d\sqrt[5]{p^4} = \sqrt[5]{\left[\tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]}$$

Consequently, if the equation

$$x^5 - Ax^3 + \frac{A^2}{5}x - D = 0$$

then each of its roots is expressed by

$$a\sqrt[5]{\left[\tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]} +$$

$$a^4\sqrt[5]{\left[\tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D - \left(\frac{A}{5}\right)^5\right]}\right]}$$

This root resembles very much, as we see, that which Cardan's formula gives for equations of the third degree. Besides this equation belongs to a peculiar class of particular equations of all degrees, the solution of which was first taught by Moivre, and of which we shall treat hereafter.

SECTION CXII.

In the same way as in § CX the equation

$$x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$$

was made rational, every other equation of the form

$$x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \dots\dots\dots + k\sqrt[n]{p^{n-1}}$$

may generally be made rational, and the degree of the rational equation will always be equal to the radical index. In this there is no other difficulty than the trouble of the calculation. Hauber has omitted this operation when $n=6$

[See the Second Collection of Combination Analytical Treatises, p. 248]. It would be desirable, if it could also be done with other values of n, because from them, as has been already shown by a few examples, the solutions of a great number of particular equations might be derived, which are so much the more worthy of observation, because they cannot be analyzed; for otherwise there could not be in the roots any radicals of the same degree as the equations themselves. Yet a great number of particular cases of the same kind may be found without such complicated calculations, by omitting at the very beginning, several terms in the general irrational expression $a\sqrt[n]{p}+b\sqrt[n]{p^2}+c\sqrt[n]{p^3}+\ldots\ldots+k\sqrt[n]{p^{n-1}}$; as the following problems will show.

SECTION CXIII.

PROB. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$ rational.

Solution. Here the two cases, in which n is an even, and the other where n is an odd number, must be distinguished.

First Case.

1. Let $x = a\sqrt[2m]{p} + b\sqrt[2m]{p^2}$; further, let α, β, γ, δ, &c. be the roots of the equation $x^{2m} - 1 = 0$. If y be substituted for $\sqrt[2m]{p}$, then the different values, which x has in relation to these irrational magnitudes, are

$$\alpha a y + \alpha^2 b y^2$$
$$\beta a y + \beta^2 b y^2$$
$$\gamma a y + \gamma^2 b y^2$$
&c.

as far as $2m$. The required rational equation is represented by

$$x^{2m} - \overset{1}{A}x^{2m-1} + \overset{2}{A}x^{2m-2} - \overset{3}{A}x^{2m-3} + \ldots\ldots$$
$$\ldots \pm \overset{m}{A}x^{m} \mp \overset{m+1}{A}x^{m-1} \pm \overset{m+2}{A}x^{m-2} \mp \ldots\ldots\ldots\ldots$$
$$\ldots - \overset{2m-1}{A}x + \overset{2m}{A} = 0$$

the upper signs obtain when m is even, and the lower when m is odd. The coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, &c. are then the sums of the former values of x, taken singly, two and two, three and three, &c.

2. The developement of these combinations gives

$$\overset{1}{A} = \grave{}[1]\,ay + \grave{}[2]\,by^2$$
$$\overset{2}{A} = \grave{}[1^2]\,a^2y^2 + \grave{}[12]\,aby^3 + \grave{}[2^2]\,b^2y^4$$
$$\overset{3}{A} = \grave{}[1^3]\,a^3y^3 + \grave{}[1^22]\,a^2by^4 + \grave{}[12^2]\,ab^2y^5 + \grave{}[2^3]\,b^3y^5$$
&c.

3. But it is evident from 2, § XCVI, that all the coefficients $\overset{1}{A}$ $\overset{2}{A}$ $\overset{3}{A}$, as far as $\overset{m}{A}$ &c. vanish, because the sum of the radical exponents in each numerical expression, is always $< 2m$, and consequently cannot be divisible by $2m$; this was evident before, from this consideration, that in the required equation there are only such powers of y, as are divisible by $2m$, for otherwise

it could not be rational. It only remains to find the coefficients $\overset{m}{A}\ \overset{m+1}{A},\ \overset{m+2}{A},\ \ldots\ldots\ \overset{2m}{A}$.

4. When in the terms of which these coefficients consist, we omit all those, which by 2, § XCVI, = o, we then find

$$\overset{m}{A} = `[2^m]\ b^m y^{2m}$$
$$\overset{m+1}{A} = `[1^2 2^{m-1}]\ a^2 b^{m-1} y^{2m}$$
$$\overset{m+2}{A} = `[1^4 2^{m-2}]\ a^4 b^{m-2} y^{2m}$$
$$\overset{m+3}{A} = `[1^6\ 2^{m-3}]\ a^6 b^{m-3} y^{2m}$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\overset{2m-1}{A} = `[1^{2m-2}2]\ a^{2m-2} b y^{2m}$$
$$\overset{2m}{B} = `[1^{2m}]\ a^{2m} y^{2m} + [2^{2m}]\ b^{2m} y^{2m}$$

5. But (§ XCVI).

$$`[2^m] \quad = \mp \frac{1\ .\ 2\ .\ 3\ \ldots\ldots\ m-1}{1\ .\ 2\ .\ 3\ \ldots\ldots\ m}\ .\ 2m = \mp 2$$

$$`[1^2 2^{m-1}] = \pm \frac{1\ .\ 2\ .\ 3 \ldots\ldots\ldots\ldots\ldots m}{1\ .\ 2\ .\ 3 \ldots m-1 \times 1\ .\ 2}.\ 2m = \pm \frac{m}{1\ .\ 2}\ .\ 2m$$

$$`[1^4 2^{m-2}] = \mp \frac{1\ .\ 2\ .\ 3 \ldots\ldots\ldots\ldots m+1}{1\ .\ 2 \ldots m-2 \times 1\ .\ 2\ .\ 3\ .\ 4} = \mp \frac{m\ .\ m^2-1}{1\ .\ 2\ .\ 3\ .\ 4}.\ 2m$$

$$`[1^6 2^{m-3}] = \pm \frac{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ldots\ldots\ m+2}{1\ .\ 2\ .\ 3\ \ldots\ m-3 \times 1\ .\ 2\ \ldots\ 6}.\ 2m =$$
$$\pm \frac{m\ .\ m^2-1\ .\ m^2-4}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6}.\ 2m$$

$$`[1^8 2^{m-4}] = \mp \frac{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ldots\ m+3}{1\ .\ 3\ .\ 3 \ldots m-4 \times 1\ .\ 2 \ldots 8}.\ 2m =$$
$$\mp \frac{m\ .\ m^2-1\ .\ m^2-4\ .\ m^2-9}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6\ .\ 7\ .\ 8}\ .\ 2m$$

. . . .

$$'[1^{2m-2}2] = +\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,2m-2}{1\times 1\,.\,2\,.\,3\,\ldots\ldots\ldots\,2m-2}.\,2m$$

$$= +\frac{m\,.\,m^2-1\,.\,m^2-4\ldots\ldots m^2-\overline{m-2}^2}{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\ldots\ldots\ldots\,2m-2}.\,2m$$

$$'[1^{2m}] \quad = -\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\,2m-1}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\ldots\,2m}.\,2m = -1$$

$$'[2^{2m}] = +\frac{(1.2.3\ldots m-1)^2}{(1.2\ldots m)^2\times 1.2}(2m)^2 - \frac{1\,.2\,.3\ldots 2m-1}{1\,.\,2\,.\,3\,\ldots 2m}\,.\,2m$$

$$= +\,2 - 1 = +\,1$$

in which the upper signs obtain when m is an even, and the lower when n is an odd number.

6. If we substitute these values in the expressions for $\overset{m}{A}, \overset{m+1}{A}, \overset{m+2}{A}$, &c., and then again put p for y^{2m}, we obtain the required rational equation

$$x^{2m} - 2b^m p x^m - \frac{m}{1\,.\,2}.\,2m\,.\,a^2 b^{m-1} p x^{m-1}$$

$$-\frac{m\,.\,m^2-1}{1\,.\,2\,.\,3\,.\,4}.\,2m\,.\,a^4 b^{m-2} p x^{m-2}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}.\,2m\,.\,a^6 b^{m-3} p x^{m-3}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m^2-9}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8.}.\,2m\,.\,a^8 b^{m-4} p x^{m-3}$$

. . . .

$$-\frac{m.m^2-1\,.\,m^2-4\,.\,m^2-9\ldots m^2-\overline{m-2}^2}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,\ldots\ldots\ldots\,2m-2}.\,2m\,.\,a^{2m-2} bpx$$

$$-\,a^{2m}p + b^{2m}p^2 = 0$$

Second Case.

7. Now, let $x=a\sqrt[2m+1]{p}+b\sqrt[2m+1]{p^2}$, or y be substituted for $\sqrt[2m+1]{p}$, $x=ay+by^2$. Further, let

$$x^{2m+1}-\overset{1}{A}x^{2m}+\overset{2}{A}x^{2m-1}-\overset{3}{A}x^{2m-2}+\ldots\ldots$$
$$\ldots\ldots\mp\overset{m+1}{A}x^{m}\pm\overset{m+2}{A}x^{m-1}\mp\overset{m+3}{A}x^{m-2}\pm\ldots\ldots$$
$$\ldots\ldots+\overset{2m}{A}x-\overset{2m+1}{A}=0$$

be the required equation, in which the upper signs obtain, when m is even, and the lower when it is odd. Then, as before, all the coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, &c. as far as $\overset{m+1}{A}$ vanish, and we have the numerical expressions referred to the equation $x^{2m+1}-1=0$,

$$\overset{m+1}{A} = `[12^m]ab^m y^{2m+1}$$
$$\overset{m+2}{A} = `[1^3 2^{m-1}]a^3 b^{m-1} y^{2m+1}$$
$$\overset{m+3}{A} = `[1^5 2^{m-2}]a^5 b^{m-2} y^{2m+1}$$
$$\overset{m+4}{A} = `[1^7 2^{m-3}]a^7 b^{m-3} y^{2m+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\overset{2m}{A} = `[1^{2m-1}2]\,a^{2m-1}by^{2m+1}$$
$$\overset{2m+1}{A} = `[1^{2m+1}]a^{2m+1}y^{2m+1}+`[2^{2m+1}]b^{2m+1}y^{2m+1}$$

8. But

$$`[12^m] \;=\pm\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\, m}{1\,.\,2\,.\,3\,\ldots\ldots\, m\times 1}.\,2m+1=\pm 2m+1$$

$$`[1^3 2^{m-1}]=\mp\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\, m+1}{1\,.\,2\,.\,3\,\ldots\ldots\, m-1\times 1.2.3}.\,2m+1$$

$$=\mp\frac{m\,.\,m+1}{1\,.\,2\,.\,3}.\,2m+1$$

$$'[1^3 2^{m-2}] = \pm \frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\, m+2}{1\,.\,2\,.\,3\,\ldots\, m-2 \times 1.2.3.4.5}\,.\,2m+1$$

$$= \pm \frac{m\,.\,m^2-1\,.\,m+2}{1\,.\,2\,.\,3\,.\,4\,.\,5}\,.\,2m+1$$

$$'[1^7 2m^{-3}] = \mp \frac{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots m+3}{1.2.3\ldots m-3 \times 1.2.3.4.5.6.7}\,.\,2m+1$$

$$= \mp \frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m+3}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}\,.\,2m+1$$

. . . .

$$'[1^{2m-1}2] = -\frac{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots 2m-1}{1 \times 1\,.\,2\,.\,3\ldots\ldots\ldots 2m-1}\,.\,2m+1$$

$$= -\frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m^2-9\ldots m^2-\overline{m-2}^{\,2}\,.\,2m-1}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots 2m-1}\,.\,2m+1$$

$$'[1^{2m+1}] = +\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\, 2m}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots 2m+1}\,.\,2m+1 = +1$$

$$'[2^{2m+1}] = +\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\, 2m}{1\,.\,2\,.\,3\ldots\ldots\ldots\ldots 2m+1}\,.\,2m+1 = +1$$

9. If these values be regularly substituted, we then obtain the required equation

$$x^{2m+1} - (2m+1)\,ab^m px^m$$

$$-\frac{m\,.\,m+1}{1\,.\,2\,.\,3}\,.\,2m+1\,.\,a^3b^{m-1}px^{m-1}$$

$$-\frac{m\,.\,m^2-1\,.\,m+2}{1\,.\,2\,.\,3\,.\,4\,.\,5}\,.\,2m+1\,.\,a^5b^{m-2}px^{-2}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m+3}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}\,.\,2m+1.a^7b^{m-3}px^{m-3}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m^2-9\,.\,m+4}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9}\,.\,2m+1.a^9b^{m-4}px^{m-4}$$

. . . .

$$-\frac{m.m^2-1.m^2-4.m^2-9\ldots m^2-\overline{m-2}^{\,2}.2m-1}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2m-1}.2m+1.a^{2m-1}bpx$$

$$-\,a^{2m+1}p - b^{2m+1}p^2 = 0$$

SECTION CXIV.

Conversely, if equations of the form in 6 and 9 of the preceding § be given, we may always find their roots. As examples, and for the sake of their use, I shall here give a few equations of this kind.

I. When $m=2$, we obtain from 9 of the preceding §, the equation

$$x^5 - 5ab^2px^2 - 5a^3bpx - a^5p - b^5p^2 = 0$$

and each of its roots

$$\alpha a\sqrt[5]{p} + \alpha^2 b\sqrt[5]{p^2}$$

when α is a root of the equation $x^5 \ldots 1 = 0$. Moreover, this equation is the same as that in I, § CXI, which was derived from the general equation of the fifth degree in § CX.

II. When $m=3$, we obtain from 6 of the preceding § the equation

$$x^6 - 2b^3px^3 - 9a^2b^2px^2 - 6a^4bpx - a^6p + b^6p^2 = 0$$

the roots of which are expressed by

$$\alpha a\sqrt[6]{p} + \alpha b\sqrt[6]{p^2}$$

when α denotes a root of the equation $x^6 - 1 = 0$.

Further, for the same value of m we obtain from 9 of the preceding §, the equation

$$x^7 - 7ab^3px^3 - 14a^3b^2px^2 - 7a^5bpx \\ - a^7p - b^7p^2 = 0$$

and each of its roots

$$\alpha a\sqrt[7]{p} + \alpha^2 b\sqrt[7]{p^2}$$

when α denotes a root of the equation $x^7 - 1 = 0$.

III. When $m=4$, we obtain from 6 of the preceding § the equation

$$x^8 - 2b^4px^4 - 16a^2b^3px^3 - 20a^4b^2px^2 \\ - 8a^6bpx - a^8p + b^8p^2 = 0$$

and for each of its roots

$$\alpha a\sqrt[8]{p} + \alpha^2 b\sqrt[8]{p^2}$$

Further, for the same value of m, we obtain from 9 the equation

$$x^9 - 9ab^4px^4 - 30a^3b^3px^3 - 27a^5b^2px^2 \\ - 9a^7bpx - a^9p - b^9p^2 = 0$$

and for each of its roots

$$\alpha a\sqrt[9]{p} + \alpha^2 b\sqrt[9]{p^2}$$

and so on.

SECTION CXV.

PROB. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$ rational.

Solution 1. If we denote the roots of the equation $x^n - 1 = 0$ by α, β, γ, δ, &c. then the roots of the required rational equation are

$$\alpha a\sqrt[n]{p} + \alpha^{n-1}b\sqrt[n]{p^{n-1}}$$
$$\beta a\sqrt[n]{p} + \beta^{n-1}b\sqrt[n]{p^{n-1}}$$
$$\gamma a\sqrt[n]{p} + \gamma^{n-1}b\sqrt[n]{p^{n-1}}$$
&c.

or since $\alpha^n = \beta^n = \gamma^n = \&c. = 1$,

$$\alpha a\sqrt[n]{p} + \alpha^{-1}b\sqrt[n]{p^{n-1}}$$
$$\beta a\sqrt[n]{p} + \beta^{-1}b\sqrt[n]{p^{n-1}}$$
$$\gamma a\sqrt[n]{p} + \gamma^{-1}b\sqrt[n]{p^{n-1}}$$
$$\&c.$$

Hence we could derive this equation in the same way as in § CXIII; the following method, however, which has been often used already, leads to the object in a shorter way.

2. Denote the sum of the first, second, third, and so on, powers of these roots by $S1$, $S2$, $S3$, &c.; then

$$S1 = {}^{\backprime}[1]a\sqrt[n]{p} + {}^{\backprime}[-1]b\sqrt[n]{p^{n-1}}$$
$$S2 = {}^{\backprime}[2]a^2\sqrt[n]{p^2} + 2{}^{\backprime}[0]abp + {}^{\backprime}[-2]b^2p^2\sqrt[n]{p^{-2}}$$
$$S3 = {}^{\backprime}[3]a^3\sqrt[n]{p^3} + 3{}^{\backprime}[1]a^2bp\sqrt[n]{p} + 3{}^{\backprime}[-1]ab^2p^2\sqrt[n]{p^{-1}}$$
$$+ {}^{\backprime}[-3]b^3p^3\sqrt[n]{p^{-3}}$$
$$S4 = {}^{\backprime}[4]a^4\sqrt[n]{p^4} + 4{}^{\backprime}[2]a^3bp\sqrt[n]{p^2} + 6{}^{\backprime}[0]a^2b^2p^2$$
$$+ 4{}^{\backprime}[-2[ab^3p^3\sqrt[n]{p^{-2}} + {}^{\backprime}[-4]b^4p^4\sqrt[n]{p^{-4}}$$
$$\&c.$$

3. From the form of these values, and from § XCVI, it follows, that the expressions $S1$, $S3$, $S5$, &c. $= 0$, and that generally each expression $S\mu = 0$, when μ is an odd number and less than n. Further, since ${}^{\backprime}[0] = n$, we have

$$S2 = \frac{2}{1}nabp$$

$$S4 = \frac{4 \,.\, 3}{1 \,.\, 2} na^2b^2p^2$$

$$S6 = \frac{6 \,.\, 5 \,.\, 4}{1 \,.\, 2 \,.\, 3} na^3b^3p^3$$

$$S8 = \frac{8 \,.\, 7 \,.\, 6 \,.\, 5}{1 \,.\, 2 \,.\, 3 \,.\, 4} na^4b^4p^4$$

and generally

$$S2\mu = \frac{2\mu \,.\, 2\mu - 1 \,.\, 2\mu - 2 \ldots\ldots \mu + 1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots \mu} na^\mu b^\mu p^\mu$$

whilst $2\mu < n$. The expression Sn, when n is an even number, besides the term

$$\frac{n \,.\, n-1 \,.\, n-2 \ldots\ldots \frac{n}{2}+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots \frac{n}{2}} na^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}}$$

which does not vanish, and which is derived from the values of $S2\mu$, when $2\mu = n$, contains also the two terms $[n]a^np$, $[-n]b^np^{n-1}$, which do not vanish, or na^np, nb^np^{n-1}. On the other hand, when n is an odd number, then the expression Sn only contains the two last-mentioned terms. Accordingly, in the case where n is an odd number, we have

$$Sn = na^np + nb^np^{n-1}$$

and in the case where n is an even number

$$Sn = \frac{n \,.\, n-1 \ldots\ldots \frac{n}{2}+1}{1 \,.\, 2 \ldots\ldots\ldots\ldots \frac{n}{2}} na^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} + na^np + nb^np^{n-1}$$

4. From the values of $S1$, $S2$, $S3$, …… Sn, already found, we are now enabled, by means of the formulæ in

§ IX (which serve for determining the coefficients of an equation from the known values of the sums of the powers of its roots), to find the coefficients of the required rational equation. Thus, if this equation be represented by

$$x^n + \overset{1}{A}x^{n-1} + \overset{2}{A}x^{n-2} + \overset{3}{A}x^{n-3} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots + \overset{m}{A}x^{n-m} + \ldots + \overset{n-1}{A}x + \overset{n}{A} = 0$$

we then obtain the following values of the assumed coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, ... $\overset{n}{A}$, in which those, whose index is odd, are omitted, because all of them (the last $\overset{n}{A}$ excepted) = 0:

$$\overset{2}{A} = -\frac{S2}{2} = -nabp$$

$$\overset{4}{A} = -\frac{\overset{2}{A}S2 + S4}{4} = +\frac{n\,.\,n-3}{1\,.\,2}a^2b^2p^2$$

$$\overset{6}{A} = -\frac{\overset{4}{A}S2 + \overset{2}{A}S4 + S6}{6} =$$
$$-\frac{n\,.\,n-4\,.\,n-5}{1\,.\,2\,.\,3}a^3b^3p^3$$

$$\overset{8}{A} = -\frac{\overset{6}{A}S2 + \overset{4}{A}S4 + \overset{2}{A}S6 + S8}{8} =$$
$$+\frac{n\,.\,n-5\,.\,n-6\,.\,n-7}{1\,.\,2\,.\,3\,.\,4}a^4b^4p^4$$

&c.

Hence the law of the progression is easily known; thus the general term is

$$\overset{2\lambda}{A} = \pm\frac{n\,.\,n-\lambda-1\,.\,n-\lambda-2 \ldots\ldots n-2\lambda+1}{1\,.\,2\,.\,3 \ldots\ldots\ldots\ldots\ldots \lambda}a^\lambda b^\lambda p^\lambda$$

The last coefficient $\overset{n}{A}$, when n is odd,

$$= -a^n p - b^n p^{n-1}$$

and when n is even is

$$\pm \frac{n \,.\, \frac{n}{2} - 1 \,.\, \frac{n}{2} - 2 \,\ldots\ldots\ldots\, 2 \,.\, 1}{1 \,.\; 2 \,.\; 3 \ldots\ldots \frac{n}{2} - 1 \,.\, \frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^{n-1}$$

or, more briefly, $\pm\, 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^n p^{n-1}$
the upper sign of the two $\pm$ obtains, when n is of the form $4m$, and the lower, when n is of the form $4m + 2$.

5. If these values be substituted in the assumed equation, we then obtain, when n is even, the required rational equation

$$\begin{aligned} x^n - \frac{n}{1} a b p x^{n-2} &+ \frac{n \,.\, n-3}{1 \,.\; 2} a^2 b^2 p^2 x^{n-4} \\ &- \frac{n \,.\, n-4 \,.\, n-5}{1 \,.\; 2 \,.\; 3} a^3 b^3 p^3 x^{n-6} \\ &+ \frac{n \,.\, n-5 \,.\, n-6 \,.\, n-7}{1 \,.\; 2 \,.\; 3 \,.\; 4} a^4 b^4 p^4 x^{n-8} \\ &- \frac{n \,.\, n-6 \,.\, n-7 \,.\, n-8 \,.\, n-9}{1 \,.\; 2 \,.\; 3 \,.\; 4 \,.\; 5} a^5 b^5 p^5 x^{n-10} \\ &\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \\ &\mp\, 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^n p^{n-1} \\ &= 0 \end{aligned}$$

the upper sign of the two $\mp$ obtains, when $n = 4m + 2$, the lower, when $n = 4m$. When n is odd, the same

equation obtains, only in the last term the magnitude $2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ must be omitted.

Example. When $n=5$, the equation is

$$x^5 - 5abpx^3 + 5a^2b^2p^2x - a^5p - b^5p^4$$

and each of its roots is

$$\alpha a\sqrt[5]{p} + \alpha^4 b\sqrt[5]{p^4}$$

Moreover, this is the same equation which was found in II, § CXI. the b in the present equation corresponding to d in the former.

II. When $n=6$, the equation is

$$x^6 - 6abpx^4 + 9a^2b^2p^2x^2 - 2a^3b^3p^3 - a^6p - b^6p^5 = 0$$

and each of its roots is

$$\alpha a\sqrt[6]{p} + \alpha^5 b\sqrt[6]{p^5}$$

Remark. Compare Michelsen's Translation of Euler's Introduction, third Book, pp. 10, 11, with this §. Euler finds the same equation, in a shorter, but less analytical way; what in his method is $\sqrt[n]{\beta}$ and α, in mine is abp and $a^np + b^np^{n-1}$. Compare also with it Huguenin's Mathematical Contributions for the further Improvement of the young Geometrician, p. 181, and so on.

If we put

$$nabp = A,\ a^np + b^np^{n-1} = T$$

then the above general equation, when n is an odd number,

is transformed into

$$x^n - Ax^{n-2} + \frac{n \, . \, n-3}{1 \, . \, 2} \; \frac{A^2}{n^2} x^{n-4} -$$
$$\frac{n : n-4 \, . \, n-5}{1 \, . \, 2 \, . \, 3} \cdot \frac{A^3}{n^3} x^{n-6} + \ldots . = T$$

the first part of this equation being continued till we come to a coefficient = o. But from the two equations $nabp = A$, $a^n p + b^n p^{n-1} = T$, we obtain

$$(a^n p - b^n p^{n-1})^2 = (a^n p + b^n p^{n-1})^2 - 4a^n b^n p^n$$
$$= T^2 - \frac{4A^n}{n^n}$$

consequently

$$a^n p - b^n p^{n-1} = \sqrt{(T^2 - \frac{4A^n}{n^n})}$$

If we combine this equation with the one $a^n p + b^n p^{n-1} = T$, we obtain, by addition and subtraction,

$$a^n p = \tfrac{1}{2} T + \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}$$

$$b^n p^{n-1} = \tfrac{1}{2} T - \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}$$

and when we extract the nth root

$$a\sqrt[n]{p} = \sqrt[n]{[\tfrac{1}{2} T + \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}]}$$

$$b\sqrt[n]{p^{n-1}} = \sqrt[n]{[\tfrac{1}{2} T - \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}]}$$

Therefore

$$a\sqrt[n]{[\tfrac{1}{2} T + \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}]} + \frac{1}{a}\sqrt[n]{[\tfrac{1}{2} T - \tfrac{1}{2} \sqrt{(T^2 - \frac{4A^n}{n^n})}]}$$

is the general expression for every root of the above equation $x^n - Ax^{n-2} + \&c. = o$.

From the resemblance of this formula to Cardan's, it follows, that the equation of Moivre is only an extension of Cardan's, and that both may be deduced in the same way, as is actually shown in the two above-mentioned works.

SECTION CXVI.

Prob. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$ rational, in the case in which n is an odd number, and not divisible by three.

Solution 1. Let

$$x^n - \overset{1}{A}x^{n-1} + \overset{2}{A}x^{n-2} + \overset{3}{A}x^{n-3} + \ldots\ldots \mp \overset{n}{A} = 0$$

be the required equation, whose roots consequently are

$$\alpha a\sqrt[n]{p} + \alpha^{n-2}b\sqrt[n]{p^{n-2}}, \text{ or } a\alpha\sqrt[n]{p} + \frac{1}{\alpha^2}bp\sqrt[n]{\frac{1}{p^2}}$$

$$\beta a\sqrt[n]{p} + \beta^{n-2}b\sqrt[n]{p^{n-2}}, \text{ or } \beta a\sqrt[n]{p} + \frac{1}{\beta^2}bp\sqrt[n]{\frac{1}{p^2}}$$

&c. &c.

Further, let the symbols $S1$, $S2$, $S3$, &c. have the same signification in reference to these roots, as in the preceding §.

2. Any undetermined power k of the first of the above roots contains, when α only is considered, the following terms:

$$\alpha^k,\ \alpha^{k-3},\ \alpha^{k-6},\ \alpha^{k-9},\ \ldots\ldots\ \alpha^{-(2k-3)},\ \alpha^{-2k}$$

the same terms contain also, with reference to β, and so on, the power k of the second root. Consequently Sk, when

α, β, γ, &c. only are considered, consists of the following terms:

$$`[k],\ `[k-3],\ `[k-6],\ \ldots\ldots\ `[-2k+3],\ `[-2k]$$

3. Now, if $k < n$, then amongst these there are no expressions but $`[o]$ and $`[-n]$, whose radical exponents are divisible by n; and indeed the first only obtains, when k is divisible by 3, but the second only when $2k-n$ is positive, and divisible by 3. But both can never occur at the same time in the same numerical expression, for otherwise, contrary to the supposition, n must be divisible by 3. Hence it follows immediately, first, that when $`[o]$ occurs in $Sk$, $k$ must be of the form $3\mu$; secondly, when $`[-n]$ occurs in Sk, the least value which k can have, is $\frac{n+3}{2}$, for which I shall substitute m; and thirdly, that then every other value of k must have the form $m+3\nu$. Consequently Sk always vanishes, when k has not one of the two forms 3μ and $m+3\nu$.

4. Now, if we seek the binomial coefficients of $`[o]$ and $`[-n]$, we obtain, when for these numerical expressions their value n is substituted,

$$S3\mu = \frac{3\mu \,.\, 3\mu-1 \,.\, 3\mu-2 \ldots . 2\mu+1}{1 \,.\, 2 \,.\, 3 \qquad \mu} n a^{2\mu} b^{\mu} p^{\mu}$$

$$S\overline{m+3\nu} = \frac{m+3\nu . m+3\nu-1 \ldots m+\nu}{1 \,.\, 2 \ldots\ldots 2\nu+1} n a^{2\nu+1} b^{m+\nu-1} p^{m+\nu-2}$$

From the first formula we obtain

$$S3 = \frac{3}{1} n a^2 b p$$

$$S6 = \frac{6 \,.\, 5}{1 \,.\, 2} na^4 b^2 p^2$$

$$S9 = \frac{9 \,.\, 8 \,.\, 7}{1 \,.\, 2 \,.\, 3} na^6 b^3 p^3$$

&c.

and from the second

$$Sm \;= \frac{m}{1} nab^{m-1} p^{m-2} = \tfrac{1}{2} \cdot \frac{n+3}{1} nab^{\frac{n+1}{2}} p^{\frac{n-1}{2}}$$

$$S\overline{m+3} = \frac{m+3 \,.\, m+2 \,.\, m+1}{1 \,.\; 2 \,.\; 3} na^3 b^m p^{m-1}$$

$$= \frac{1}{2^3} \cdot \frac{n+9 \,.\, n+7 \,.\, n+5}{1 \,.\; 2 \,.\; 3} na^3 b^{\frac{n+3}{2}} p^{\frac{n+1}{2}}$$

$$S\overline{m+6} = \frac{m+6 \,.\, m+5 \,.\, m+4 \,.\, m+3 \,.\, m+2}{1 \,.\; 2 \,.\; 3 \,.\; 4 \,.\; 5} na^5 b^{m+1} p^m$$

$$= \frac{1}{2^5} \cdot \frac{n+15 \,.\, n+13 \,.\, n+11 \,.\, n+9 \,.\, n+7}{1 \,.\; 2 \,.\; 3 \,.\; 4 \,.\; 5} na^5 b^{\frac{n+5}{2}} b^{\frac{n+3}{2}}$$

&c.

Every other Sk, which is not included amongst these, $=$ o.

5. Hence the coefficients of the required equation may be determined by means of the general formula

$$\pi \overset{\pi}{A} = \overset{\pi-1}{A} S1 - \overset{\pi-2}{A} S2 + \overset{\pi-3}{A} S3 - \overset{\pi-4}{A} S4 + \ldots\ldots$$
$$\ldots\ldots\ldots \mp \overset{1}{A} S\overline{\pi - 1} \pm S\pi$$

When in this we substitute 1, 2, 3, 4, 5, &c. successively for π, it will immediately be shown, that all the coefficients, those excepted which are under the forms $\overset{3\mu}{A}$, $\overset{m+3\mu}{A}$, vanish, because in the products, of which this formula is compounded, either a coefficient, or a numerical expression, or even both at once vanish. Further, we find

$$\overset{3}{A} = \frac{S3}{3} = na^2bp$$

$$\overset{6}{A} = \frac{\overset{6}{A}S3 - S6}{6} = \frac{n \,.\, n-5}{1 \,.\, 2} a^4b^2p^2$$

$$\overset{9}{A} = \frac{\overset{6}{A}S3 - \overset{3}{A}S6 + S9}{9} = \frac{n \,.\, n-7 \,.\, n-8}{1 \,.\, 2 \,.\, 3} a^6b^3p^3$$

&c.

$$\overset{m}{A} = \mp\frac{Sm}{m} = \mp\, nab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}$$

$$\overset{m+3}{A} = \frac{\overset{m}{A}S3 \mp \overset{3}{A}Sm}{m+3} = \frac{1}{2^2}\cdot\frac{n \,.\, n-5 \,.\, n-7}{1 \,.\, 2 \,.\, 3} a^3b^{\frac{n+3}{2}}p^{\frac{n+1}{2}}$$

$$\overset{m+6}{A} = \frac{\overset{m+3}{A}S3 - \overset{m}{A}S6 \mp \overset{6}{A}Sm \pm \overset{3}{A}S\overline{m+3} \mp S\overline{m+6}}{m+6}$$

$$= \mp\frac{1}{2^4}\cdot\frac{n \,.\, n-7 \,.\, n-9 \,.\, n-11 \,.\, n-13}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} a^5b^{\frac{n+5}{2}}p^{\frac{n+3}{2}}$$

the upper signs obtain when m is even, the lower when n is odd.

6. The last term $\overset{n}{A}$ can neither be included in the form $\overset{3\mu}{A}$, nor in the form $\overset{m+3\mu}{A}$, because otherwise n must be divisible by 3. But this term can be very easily found by taking the product of all the roots in 1, which product is here reducible merely to the sums of the products of all its first and all its second parts, and consequently $= a^{n}p + b^{3n}p^{n-6}$.

7. If we substitute these values of the coefficients in the assumed equation in 1, we then obtain the required rational equation

$$x^n - \frac{n}{1}a^2bpx^{n-3} + \frac{n \,.\, n-5}{1 \,.\, 2}a^4b^2p^2x^{n-6}$$

$$-\frac{n \,.\, n-7 \,.\, n-8}{1 \,.\, 2 \,.\, 3}a^6b^3p^3x^{n-9}$$

$$+\frac{n \,.\, n-9 \,.\, n-10 \,.\, n-11}{1 \,.\, 2 \,.\, 3 \,.\, 4}a^8b^4p^4x^{n-12}$$

$$-\frac{n \,.\, n-11 \,.\, n-12 \,.\, n-13 \,.\, n-14}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5}a^{10}b^5p^5x^{n-15}$$

$$+\frac{n \,.\, n-13 \,.\, n-14 \,.\, n-15 \,.\, n-16 \,.\, n-17}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6}a^{12}b^6p^6x^{n-18}$$

&c.

$$-\frac{n}{1}ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}x^{\frac{n-3}{2}}$$

$$-\frac{1}{2^2} \cdot \frac{n \,.\, n-5 \,.\, n-7}{1 \,.\, 2 \,.\, 3}a^3b^{\frac{n+3}{2}}p^{\frac{n+1}{2}}x^{\frac{n-9}{2}}$$

$$-\frac{1}{2^4} \cdot \frac{n \,.\, n-7 \,.\, n-9 \,.\, n-11 \,.\, n-13}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5}a^5b^{\frac{n+5}{2}}p^{\frac{n+3}{2}}x^{\frac{n-15}{2}}$$

$$-\frac{1}{2^6} \cdot \frac{n \,.\, n-9 \,.\, n-11 \,.\, n-13 \,.\, n-15 \,.\, n-17 \,.\, n-19}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7}a^7b^{\frac{n+7}{2}}p^{\frac{n+5}{2}}x^{\frac{n-21}{2}}$$

&c.

$$-a^np - b^np^{n-2}$$

$$= 0.$$

The two series in this equation are continued till we arrive at the negative exponents of x.

Example I. When $n=5$, we find the equation

$$x^5 - 5a^2bpx^2 - 5ab^3p^2x - a^5p - b^5p^3 = 0$$

and for each of its roots

$$a\alpha\sqrt[3]{p} + \alpha^3b\sqrt[5]{p^3}$$

If we put $5a^2bp=B$, $5ab^3p^2=C$, then this equation is

transformed into the following one:

$$x^5 - Bx^2 - Cx - \frac{B^3}{25C} - \frac{C^2}{5B} = 0$$

the same equation, which was found in I, § CXI.

EXAMPLE II. When $n=7$, we obtain the equation

$$x^7 - 7a^2bpx^4 - 7ab^4p^3x^2 + 7a^4b^2p^2x - a^7p - b^7p^5 = 0$$

and each of its roots is

$$\alpha a\sqrt[7]{p} + \alpha^5 b\sqrt[7]{p^5}$$

To this belong all equations of the seventh degree of the form

$$x^7 - 7Ax^4 - 7Bx^2 + 7A^2x - \frac{A^4}{B} - \frac{B^2}{A} = 0$$

as may be easily found, by putting $a^2bp = A$, $ab^4p^3 = B$.

SECTION CXVII.

In the same way as in § CXIII, § CXV, and § CXVI, we can derive innumerable other general forms of solvible equations from the binomial $\sqrt[n]{p^\mu} + \sqrt[n]{p^\nu}$ by eliminating the irrational magnitudes. However, since otherwise this subject possesses no interest, I shall here content myself with giving an equation, which Waring, one of the most celebrated Analysts that England ever possessed, gives in a treatise on the general solution of equations, by assuming $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$ (Philosophical Transactions for the year 1779, p. 92). When n is odd, this equation is

$$x^n - p\,[na^{n-3}bx^2 + \frac{n\,.\,n-5}{1\,.\,2}a^{n-6}b^2x^4 +$$

$$\frac{n\,.\,n-7\,.\,n-8}{1\,.\,2\,.\,3}a^{n-9}b^3x^6 + \frac{n.n-9.n-10.n-11}{1\,.\,2\,.\,3\,.\,4}a^{n-12}b^3x^8$$

$$+ \frac{n\,.\,n-11\,.\,n-12\,.\,n-13\,.\,n-14}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{n-15}b^5x^{10} + \&c.]$$

$$\pm\, p^2[na^{\frac{n-3}{2}}b^{\frac{n+1}{2}}x - \frac{1}{2^2}\cdot\frac{n\,.\,n-5\,.\,n-7}{1\,.\,2\,.\,3}a^{\frac{n-1}{2}}b^{\frac{n+3}{2}}x^3$$

$$+ \frac{1}{2^4}\cdot\frac{n\,.\,n-7\,.\,n-9\,.\,n-11\,.\,n-13}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{\frac{n-15}{2}}b^{\frac{n+5}{2}}x^5 -$$

$$\frac{1}{2^6}\,\frac{n.n-9.n-11.n-13.n-15.n-17.n-19}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}a^{\frac{n-21}{2}}b^{\frac{n+7}{2}}x^7 + \&c.]$$

$$- a^np - b^np^3 = 0.$$

The factor p^2 has the sign $+$, when $\frac{n-3}{4}$ is a whole number, but in other cases the sign $-$.

SECTION CXVIII.

The method by which equations of the first degree are made rational, which hitherto we have chiefly used for finding solvable equations, can also be used with advantage when we merely wish to clear the unknown magnitudes in an equation of the radical sign. I shall elucidate this by two examples.

Let the equation

$$p + \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \sqrt[\varrho]{t} \,\ldots\ldots = 0$$

be given, and let p, q, r, s, &c. be rational functions of the unknown magnitudes y, z, &c.: it is required to make this equation rational.

Put $-p = x$; then we have in reference to x the equation of the first degree, viz.

$$x = \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \sqrt[\rho]{t} \ldots\ldots$$

We try to make this equation rational, and then again put $-p$ for x, we then have the required equation clear of irrational magnitudes.

If the equation

$$\sqrt[\mu]{p} + \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} \ldots\ldots = 0$$

be given, in which there is not even one rational term, we put $-\sqrt[\mu]{p} = x$, and then make the equation $x = \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} + \ldots\ldots$ rational. Having done this, we substitute in the obtained equation for x successively its values $-\alpha\sqrt[\mu]{p}$, $-\beta\sqrt[\mu]{p}$, $-\gamma\sqrt[\mu]{p}$, &c.; then there arise μ equations. If we multiply these together, we then obtain the required equation clear of irrational magnitudes.

On the subject of clearing equations of irrational magnitudes, there is a very able treatise by Fischer, in the Hindenburg Archives of Mathematics, Number VIII, written with that clearness and perspicuity peculiar to the author.

The rational equation which is obtained from the equation $x = \sqrt[\mu]{p} + \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} + \ldots$ so long as the irrational magnitudes $\sqrt[\mu]{p}$, $\sqrt[\nu]{q}$, $\sqrt[\pi]{r}$, $\sqrt[\rho]{s}$, &c. have no particular relation to each other, always rises to the degree $\mu\nu\pi\rho\ldots$, because the μ values of $\sqrt[\mu]{p}$, the ν vales of $\sqrt[\nu]{q}$, the π values of $\sqrt[\pi]{r}$, and so on, may be combined

together exactly this number of times. But if the above irrational magnitudes have any relation to each other, so that if one or other is determined, the remaining ones are either all, or only in part determined, then the rational equation is always of a lower degree. Here follow a few examples by way of elucidation.

The rational equation for $x = \sqrt[n]{p} + \sqrt[n]{p^2} + \sqrt[n]{p^3} + \sqrt[n]{p^4} + \ldots + \sqrt[n]{p^t}$ only rises to the nth degree, because $\sqrt[n]{p^2} = (\sqrt[n]{p})^2$, $\sqrt[n]{p^3} = (\sqrt[n]{p})^3$, $\sqrt[n]{p^4} = (\sqrt[n]{p})^4$, &c. and consequently x contains no more values than the irrational magnitudes $\sqrt[n]{p}$.

The rational equation for $x = \sqrt[12]{p} + \sqrt[3]{p} + \sqrt[5]{q}$ only rises, but necessarily, to the sixtieth degree, because $\sqrt[3]{p} = (\sqrt[12]{p})^4$, and the twelve values of $\sqrt[12]{p}$, combined with the five values of $\sqrt[5]{q}$, gives sixty different values of x.

The rational equation for $x = \sqrt[12]{p^5} + \sqrt[9]{p^7} + \sqrt[8]{p^3} = p^{\frac{5}{12}} + p^{\frac{7}{9}} + p^{\frac{3}{8}}$ only rises to the seventy-second degree. For if we reduce the small fractional exponents to the least common denominator, we have $x = p^{\frac{30}{72}} + p^{\frac{56}{72}} + p^{\frac{27}{72}}$. If $\therefore \alpha$ be a root of the equation $y^{72} - 1 = 0$, then is $\alpha^{30} p^{\frac{30}{72}} + \alpha^{56} p^{\frac{56}{72}} + \alpha^{27} p^{\frac{27}{72}}$ or $\alpha^{30} \sqrt[12]{p^5} + \alpha^{56} \sqrt[9]{p^7} + \alpha^{27} \sqrt[8]{p^3}$ the corresponding value of x, and there are seventy-two of these values.

SECTION CXIX.

PROB. Find a factor, by which the given irrational expression $p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots\ldots$ must be multiplied, in order to make it rational.

Solution. Let a, b, c, d, &c. denote the different values which this expression has, when the irrational magnitudes are taken in all possible ways. Now, if we form the equation $x=p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots\ldots$, then the rational equation derived from it, is

$$(x-a)\ (x-b)\ (x-c)\ (x-d)\ \ldots\ldots=0$$

and its last term $=\pm\, a\, b\, c\, d\ \ldots\ldots$ Now, since this product must be rational, it follows that the given expression is rational, when we multiply it by the product of all the remaining expressions which its different values give; and this product consequently is the factor sought.

EXAMPLE I. Let the expression $p+\sqrt[3]{q}$ be given. If then 1, α, β, be the three roots of the equation $y^3-1=0$, then the required factor

$$=(p+\alpha\sqrt[3]{q})\ (p+\beta\sqrt[3]{q})$$

or, since $\alpha+\beta=-1$, $\alpha\beta=1$,

$$=p^2-p\sqrt[3]{q}+\sqrt[3]{q^2}$$

EXAMPLE II. For the expression $p+\sqrt{q}+\sqrt{r}$, the required factor

$$= (p+\sqrt{q}-\sqrt{r})\ (p-\sqrt{q}+\sqrt{r})\ (p-\sqrt{q}-\sqrt{r})$$
$$= p^3 - pq - pr - (p^2 - q + r)\sqrt{q} -$$
$$(p^2 + q - r)\sqrt{r} + 2p\sqrt{qr}$$

which is already known.

EXAMPLE III. For the expression $p+\sqrt[3]{q}-\sqrt[4]{r}$, when $1, \alpha, \beta$, are the roots of the equation $y^3-1=o$, we obtain the following factor:

$$(p+\sqrt[3]{q}+\sqrt[4]{r})\ (p+\alpha\sqrt[3]{q}+\sqrt[4]{r})\ (p+\beta\sqrt[3]{q}+\sqrt[4]{r})$$
$$(p+\sqrt[3]{q}+\sqrt[4]{r}\,.\,\sqrt{-1})\ (p+\alpha\sqrt[3]{q}+\sqrt[4]{r}\,.\,\sqrt{-1})$$
$$(p+\beta\sqrt[3]{q}+\sqrt[4]{r}\,.\,\sqrt{-1})\ (p+\sqrt[3]{q}-\sqrt[4]{r}\,.\,\sqrt{-1})$$
$$(p+\alpha\sqrt[3]{q}-\sqrt[4]{r}\,.\,\sqrt{-1})\ (p+\beta\sqrt[3]{q}-\sqrt[4]{r}\,.\,\sqrt{-1})$$
$$(p+\alpha\sqrt[3]{q}-\sqrt[4]{r})\ (p+\beta\sqrt[3]{q}-\sqrt[4]{r})$$

Remark. When $p=o$, a more simple factor may often be found, by which the object to make the given expression rational may be attained. Thus, if the expression $\sqrt[3]{q}+\sqrt[3]{r}$ be already rational, when we multiply it only by the factor $(\sqrt[3]{q} + \alpha\sqrt[3]{r})\ (\sqrt[3]{q} + \beta\sqrt[3]{r}) = \sqrt[3]{q^2} -\sqrt[3]{qr}+\sqrt[3]{r^2}$; and the expression $\sqrt{q}+\sqrt{r}+\sqrt{s}$ is so by multiplying by the factor $(\sqrt{q} + \sqrt{r} - \sqrt{s})$ $(\sqrt{q}-\sqrt{r}+\sqrt{s})$ $(\sqrt{q}-\sqrt{r}-\sqrt{s})$. This is always the case, when the indices of the roots have a common divisor, or the irrational magnitudes have a certain relation to each other.

Corollary. From what has been already said, it follows, that it is always possible, by the multiplication of the

numerator and denominator of a given fraction by a proper factor, to clear the denominator of irrational magnitudes. Consequently also an equation of the form

$$x=\frac{p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots}{p'+\sqrt[\mu']{q'}+\sqrt[\nu']{r'}+\sqrt[\pi']{s'}+\ldots}$$ may always be reduced

to an equation of the form $x=p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots$ and as one of this kind can always be made rational by the preceding §, so in like manner the former may always be made rational.

SECTION CXX.

Prob. Make the equation $x = \sqrt{p} + \sqrt{q}$ rational, when the magnitudes p and q are not immediately given, but only assumed to be the roots of an equation of the second degree $y^2 - Ay + B = 0$.

Solution. The values which x has by the different determination of its irrational magnitudes, are

$$\sqrt{p} + \sqrt{q}, \quad -\sqrt{p} + \sqrt{q}$$
$$\sqrt{p} - \sqrt{q}, \quad -\sqrt{p} - \sqrt{q}$$

The mere inspection of these values shows, that when we substitute p and q for each other, they undergo no further change, than that one is transformed into the other. Consequently in the rational equation derived from $x = \sqrt{p} + \sqrt{q}$ there is no change, when we substitute p for q, and it must ∴ necessarily be a symmetrical function of these magnitudes, and consequently may be expressed rationally by the coefficients A, B. If ∴ we eliminate

p and q by means of the given coefficients A, B, we obtain the required rational equation.

But from $x = \sqrt{p} + \sqrt{q}$ we obtain the equation (§ XCVIII.)

$$x^4 - 2(p + q)x^2 + (p + q)^2 = 0$$

or

$$x^4 - 2(p + q)x^2 + (p + q)^2 - 4pq = 0$$

If in the latter equation we substitute for $p+q$ and pq their values A and B, we obtain the required equation

$$x^4 - 2Ax^2 + A^2 - 4B = 0$$

SECTION CXXI.

Prob. Make the equation

$$x = \sqrt[n]{p} + \sqrt[n]{q} + \sqrt[n]{r} + \sqrt[n]{s} + \ldots.$$

rational; when the m magnitudes p, q, r, s, &c. are not immediately given, but only assumed to be the roots of a given equation of the mth degree

$$y^m - Ay^{m-1} + By^{m-2} - Cy^{m-3} + \text{\&c.} = 0$$

Solution. If we endeavour to find all the possible values of x, which arise from the different combinations of the values of the roots, and then in these put the magnitudes p, q, r, s, &c. for one another in any way, but let it be the same one in all the values, the consequence will only be this, that these values either undergo no change, or merely that one is transformed into another. For let α, β, γ, δ, &c. be the roots of the equation $x^n - 1 = 0$, and $\alpha\sqrt[n]{p}+\beta\sqrt[n]{q}+\gamma\sqrt[n]{r}+\delta\sqrt[n]{s}+$&c. any value of x. Now, if it be possible, that from this expres-

sion, by any substitution of the magnitudes p, q, r, s, &c. for one another, viz. by that of p for q, another expression $\alpha\sqrt[n]{q}+\beta\sqrt[n]{p}+\gamma\sqrt[n]{r}+\delta\sqrt[n]{s}+$&c. is generated, which does not belong to the values of x; then there must be, contrary to the supposition, a combination of the values of the roots, which is not included in the values of x. Since $\therefore$ the values of x remain the same, however the magnitudes p, q, r, s, &c. are subtituted for one another, then must the rational equation derived from $x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+$&c. be a symmetrical function of these magnitudes, and consequently may be expressed rationally by A, B, C, &c. the coefficients of the given equation. If $\therefore$ we eliminate the magnitudes p, q, r, s, &c. by means of these coefficients, we obtain the required equation.

Corollary. The equation, which we obtain under the condition of the problem, is consequently always of the same degree as the equation derived from $x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+$&c. when the magnitudes p, q, r, s, &c. are independent of each other. In the latter case, however, the rational equation is of the degree $nnnn\ldots=n^m$, when m denotes the number of the magnitudes p, q, r, s, &c. (§ CXVIII), consequently also in the former.

SECTION CXXII.

Rule The rational equation for

$$x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+\&c.$$

in the case in which the m magnitudes p, q, r, s, &c. are either wholly independent of each other, or the roots of an equation of the mth degree, can only contain such powers of x, whose exponents are divisible by n.

PROOF. Let

$$x^k + \overset{1}{A}x^{k-1} + \overset{2}{A}x^{k-2} + \ldots\ldots + \overset{\mu}{A}x^{k-\mu} + \ldots = 0$$

be the equation, which arises from the multiplication of all the positive distinct equations of the form

$$x' - \alpha\sqrt[n]{p} - \beta\sqrt[n]{q} - \gamma\sqrt[n]{r} - \delta\sqrt[n]{s} - \ldots\ldots = 0$$

where α, β, γ, δ, &c. denote the roots of the equation $x^n - 1 = 0$; now it is evident, that the undetermined coefficient $\overset{\mu}{A}$ can contain no other numerical expressions of these roots, but those in which the sum of the radical exponents $= \mu$. Now, since these always vanish, when μ is not divisible by n (§ XCVI. 2), then likewise must the term $\overset{\mu}{A}x^{k-\mu}$ always vanish. But when μ is not divisible by n, then also $k-\mu$ cannot be divisible by n, because $k = n^m$; $\therefore$ all those terms vanish which contain $k-\mu$, consequently the exponent of x is not divisible by n. Therefore the rational equation contains those terms only, in which the exponent of x is divisible by n. *Q. E. D.*

SECTION CXXIII.

PROB. Let the m magnitudes y, z, t, u, &c. be given by the m equations

$$y^{\mu} + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0$$
$$z^{\nu} + A'z^{\nu-1} + B'z^{\nu-2} + C'z^{\nu-3} + \&c. = 0$$
$$t^{\pi} + A''t^{\pi-1} + B''t^{\pi-2} + C''t^{\pi-3} + \&c. = 0$$
$$\&c.$$

consequently irrational: required to make the equation

$$x = y + z + t + u + \&c.$$

rational.

Solution. Since y is given by an equation of the μth degree, z by an equation of the νth degree, and so on, we have μ values for y, ν values for z, and so on. If we combine all these values in as many ways as possible, to the number $y + z + t + u +$ &c., we find all the values of x, and consequently by the multiplication of all the distinct equations of the form $x - (y+z+t+u+\&c.) = 0$ the required equation, which necessarily is of the degree $\mu\nu\pi\ldots$, because the different values of y, z, t, u, &c. may be combined in this number of ways. Now, I assert, that with respect to the different values of y, which may be y', y'', y''', &c. this equation is symmetrical. For since the values of x in the substitution of these magnitudes, undergo no further change, than that one is transformed into another, consequently, also the equation itself must be such, that it undergoes no change in the substitution of the magnitudes, y', y'', y''', &c. Therefore, the magnitudes y', y'', y''', &c. may be eliminated by means of the coefficients A, B, C, &c, What has been said here of y and its different values y', y'', y''', &c. applies also to z, t, &c. and its values z', z'', z''', &c. t', t'', t''', &c. &c.; and consequently these

magnitudes may also be eliminated by means of the coefficients A', B', C', &c. A'', B'', C'', &c. &c. In this way we likewise obtain a rational equation for x, which only contains known magnitudes; and this is the equation sought.

Example. Let $x=y+z$; let the magnitudes y and z be given by the two equations

$$y^3 - Ay^2 + By - C = 0$$
$$z^2 - A'z + B' = 0$$

required to find the rational equation for x.

The different values of x are

$$\begin{array}{l} y' + z', \; y' + z'' \\ y'' + z', \; y'' + z' \\ y''' + z', \; y''' + z'' \end{array}$$

Denote the sum of the first, second, third, &c. powers of these roots by $S1$, $S2$, $S3$, &c. we then have

$$S1 = 2(y'+y''+y''') + 3(z'+z'') = 2A + 3A'$$
$$S2 = 2(y'^2+y''^2+y'''^2) + 2(y'+y''+y''')(z'+z'') + 3(z'^2+z''^2)$$
$$= 2(A^2-2B) + 2AA' + 3(A'^2-2B')$$
$$S3 = 2(y'^3+y''^3+y'''^3) + 3(y'^2+y''^2+y'''^2)(z'+z'') + 3(y'+y''+y''')(z'^2+z''^2) + 3(z'^3+z''^3)$$
$$= 2(A^3-3AB+3C) + 3(A^2-2B)A' + 3A(A'^2-2B') + (3A'^3-3A'B')$$

&c.

Thus having calculated the values of $S1$, $S2$, $S3$, &c. when the required equation is represented by

$$x^6 - \overset{1}{A}x^5 + \overset{2}{A}x^4 - \overset{3}{A}x^3 + \overset{4}{A}x^2 - \overset{5}{A}x + \overset{6}{A} = 0$$

we then obtain its coefficients, by means of the equations

$$\overset{1}{A} = S1$$

$$\overset{2}{A} = \frac{\overset{1}{A}S1 - S2}{2}$$

&c.

SECTION CXXIV.

PROB. Let $f:\ (y)\ (z)\ (t)\ (u) \ldots\ldots$ denote any rational function of the magnitudes $y,\ z,\ t,\ u$, &c., which are represented by the same number of equations

$$y^{\mu} + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \text{\&c.} = 0$$
$$z^{\nu} + Bz^{\nu-1} + Bz^{\nu-2} + Cz^{\nu-3} + \text{\&c.} = 0$$
$$t^{\pi} + At^{\pi-1} + Bt^{\pi-2} + Ct^{\pi-3} + \text{\&c.} = 0$$

find an equation for the values of this function.

Solution. Put $x = f:\ (y)\ (z)\ (t)\ (u) \ldots$, find all the possible values of this function, and from these form the equation for x: then eliminate the values y', y'', y''', &c. of y, by means of the coefficients A, B, C, &c. of that equation by which this magnitude is given; which may always be done, because the equation for x must necessarily be a symmetrical function of the magnitudes y', y'', y''', &c. If we proceed with z, t, u, &c. in the same way as we did with y, we obtain an equation for x, whose coefficients are all known; and this is the required equation.

Corollary. In order to find all the values of x, we must combine the μ values of y, the ν values of z, and

so on, in all possible ways in the function f: (y) (z) (t) (u)... But it is evident, that the number of these combinations $= \mu\nu\pi...$; consequently also this product gives the number of the values of x, and $\therefore$ the degree of the transformed equation.

If the function only contains the magnitude y, or if $x = f$: (y), then the equation for y is only of the μth degree, consequently in this case, the transformed equation is of the same degree, as the equation by which y is given.

SECTION CXXV.

Prob. The unknown magnitude x is given by the equation of the nth degree, viz.

$$x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \&c. = 0$$

the coefficients P, Q, &c. however, are not given immediately, but merely all assumed to be known functions of a magnitude y, which depends on an equation of the nth degree

$$y^\mu + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0$$

required to find an equation for x, which only contains known magnitudes.

Solution. Denote the roots of the equation $y^\mu + Ay^{\mu-1} + \&c. = 0$ by y', y'', y''', &c., and introduce these values into the functions P, Q, &c. Now if we denote that, into which these functions are transformed, when we substitute in them y', y'', y''', &c. successively for y, by P', Q', &c., P'', Q'', &c. P''', Q''', &c. &c.; we then

obtain the following μ equations, all of which must obtain at the same time:

$$x^n + P'x^{n-1} + Q'x^{n-2} + R'x^{n-3} + \&c. = o$$
$$x^n + P''x^{n-1} + Q''x^{n-2} + R''x^{n-3} + \&c. = o$$
$$x^n + P'''x^{n-1} + Q'''x^{n-2} + R'''x^{n-3} + \&c. = o$$
$$\&c.$$

The product of these $\therefore$ gives the required equation.

But since these equations are such, that in the transposition of the magnitudes y', y'', y''', &c. no other change takes place, but that one is merely transformed to another; consequently their product suffers no change in their transposition, and $\therefore$ with reference to y', y'', y''', &c. they are symmetrical. Therefore these magnitudes may always be eliminated by means of the given coefficients A, B, C, &c.

Corollary. But conversely, every equation of the $n\mu$th degree, which can be considered as arising from the elimination of y in the two equations $x^n + Px^{n-1} + \&c. = o$, $y^\mu + Ay^{\mu-1} + \&c. = o$, may always be solved, if the solution of equations of the nth and μth degrees be presupposed; for the second equation gives the value of y, and when we substitute this value in the first equation, then the latter gives the value of x.

SECTION CXXVI.

PROB. The unknown magnitude x is given by the equation of the nth degree, viz.

$$x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \&c. = o$$

It is assumed, that the coefficients P, Q, R, &c. are functions of the magnitudes y, z, t, u, &c., and that these magnitudes are given by the equations

$$y^{\mu} + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0$$
$$z^{\nu} + A'z^{\nu-1} + B'z^{\nu-2} + C'z^{\nu-3} + \&c. = 0$$
$$t^{\pi} + A''t^{\pi-1} + B''t^{\pi-2} + C''t^{\pi-3} + \&c. = 0$$
$$\&c.$$

Find an equation for x, which contains known magnitudes only.

Solution. In the first place we consider P, Q, R, &c. merely as functions of y, and eliminate these magnitudes by the method in the preceding §; then we obtain an equation for x, of the $n\mu$th degree, which only contains the unknown magnitudes z, t, u, &c. If in the same way we also eliminate the magnitudes z, t, u, &c. successively, we shall at length get an equation of the $n\mu\nu\pi$...th degree, which only contains x and the known magnitudes A, B, C, &c. A', B', C', &c., A'', B'', C'', &c. &c. and which consequently is the equation sought.

VI.—General inquiries respecting the transformation of equations.

SECTION CXXVII.

AT the end of the fourth chapter the transformation of equations and Tschirnhausen's method were mentioned, and its application to the solution of equations of the third and fourth degrees. This is now the proper place to give some deeper inquiries respecting it, in order that we may ascertain, what may be expected of this method in its application to equations of higher degrees.

SECTION CXXVIII.

Prob. Let

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

be the given equation, and

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots + kx + l = y$$

the auxiliary one; consequently the transformed equation for y is of the nth degree (§ LXXXI), viz.

$$y^n + \mathfrak{A}y^{n-1} + \mathfrak{B}y^{n-2} + \mathfrak{C}y^{n-3} + \&c. = 0$$

Show in what dimension the coefficients a, b, c, &c. of the assumed equation enter into the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

Solution. If x', x'', x''', &c. denote the roots of the given equation, then the transformed equation $y^n + \mathfrak{A}y^{n-1} + \&c. = 0$ has the following roots:

$$\begin{aligned} y' &= x'^m + ax'^{m-1} + bx'^{m-2} + \ldots\ldots + kx' + l \\ y'' &= x''^m + ax''^{m-1} + bx''^{m-2} + \ldots\ldots + kx'' + l \\ y''' &= x'''^m + ax'''^{m-1} + bx'''^{m-2} + \ldots\ldots + kx''' + l \\ &\&c. \end{aligned}$$

Now, since the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. are the sums of these roots taken singly, two and two, three and three, and so on, we may ∴ conclude, that the letters a, b, c, &c. occur in $\mathfrak{A}$ in the first dimension, in $\mathfrak{B}$ in the second, in $\mathfrak{C}$ in the third, and, generally, in the pth coefficient in the pth power.

SECTION CXXIX.

PROB. Required to find of what degree the auxiliary equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots + kx + l = y$$

must be, when it is possible to transform the general equation of the nth degree

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

into an equation of two terms of the form

$$y^n - V = 0$$

Solution. In the equation $y^n - V = 0$, $n - 1$ terms are wanting; ∴ the auxiliary equation contains as many undetermined magnitudes a, b, c, &c. by determining which, we are enabled to eliminate these terms; it must consequently be of the $n-1$th degree, and ∴ $m = n - 1$.

SECTION CXXX.

PROB. To transform the given equation $x^3 - Ax^2 + Bx - C = o$ into one of two terms $y^3 - V = o$, we must assume the auxiliary equation $x^2 + ax + b = y$ (preceding §): determining, a priori, the degree of the equations, on which the coefficients a, b, depend.

Solution 1. If we denote one of the two primitive roots of the equation $y^3 - 1 = o$ by α, and put $\sqrt[3]{V} = y'$, then y', $\alpha y'$, $\alpha^2 y'$ are the three roots of the equation $y^3 - V = o$. Each of the three roots x', x'', x''', of the given equation corresponds to one of the values of y: which? It remains undetermined.

2. If we combine the values of x in all possible ways with the values of y, we then obtain the six following combinations:

$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x', & x'', & x''' \end{pmatrix}; \quad \begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x', & x''', & x'' \end{pmatrix}$$
$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x'', & x', & x''' \end{pmatrix}; \quad \begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x'', & x''', & x' \end{pmatrix}$$
$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x''', & x', & x'' \end{pmatrix}; \quad \begin{pmatrix} y', & y', & \alpha^2 y' \\ x''', & x'', & x' \end{pmatrix}$$

3. If the values of x and y, which correspond to each other in the first combination, be substituted in the auxiliary equation, we obtain the three equations:

$$\begin{aligned} x'^2 + ax' + b &= y' \\ x''^2 + ax'' + b &= \alpha y' \\ x'''^2 + ax''' + b &= \alpha^2 y' \end{aligned}$$

and by means of these equations the values of a and b may be determined from x', x'', x'''. In the first place, in order to determine a, multiply the second equation by α, and the third by α^2, and then add it to the first; thus, since $\alpha^4 = \alpha^3 . \alpha = \alpha$, and $1 + \alpha + \alpha^2 = {}^{\backprime}[1] = 0$,

$$x'^2 + \alpha x''^2 + \alpha^2 x'''^2 + a\,(x' + \alpha x'' + \alpha^2 x''') = 0$$

we obtain

$$a = -\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}$$

4. From the remaining five combinations in 2, there may be found five other values of a. To effect this, however, it is not necessary to begin the calculation anew: for since the above combinations only differ in this, that the roots x', x'', x''' are transposed, we can even take the transposition found in the expression for a. By these means we obtain the six following values.

$$\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}, \quad \frac{x'^2 + \alpha x'''^2 + \alpha x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$\frac{x''^2 + \alpha x'''^2 + \alpha^2 x'^2}{x'' + \alpha x''' + \alpha^2 x'}, \quad \frac{x'''^2 + \alpha x''^2 + \alpha^2 x'^2}{x''' + \alpha x'' + \alpha^2 x'}$$

$$\frac{x'''^2 + \alpha x'^2 + \alpha^2 x''^2}{x''' + \alpha x' + \alpha^2 x''}, \quad \frac{x''^2 + \alpha x'^2 + \alpha^2 x'''^2}{x'' + \alpha x' + \alpha^2 x'''}$$

5. But of these six values, only the two first, which are opposite to each other, are actually different. For if in these two values we multiply both the numerator and denominator by α^2, we then obtain the next two; and from these again the two last, by multiplying the numerator and denominator by α^2

6. The coefficient a .·. has only the two different values

$$\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}, \quad \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

and consequently it depends merely on an equation of the second degree.

7. Further, if we add the three equations in 3 together, we obtain

$$x'^2 + x''^2 + x'''^2 + a\,(x' + x'' + x''') + 3b = 0$$

or

$$[2] + a\,[1] + 3b = 0,$$

and when in this we substitute for a its two values, we then also obtain for b two values, and consequently this coefficient also depends on an equation of the second degree.

Corollary. If we wish actually to find the equation for a, then let

$$a^2 - Pa + Q = 0.$$

The roots of this equation, then, are the two which were found in 6, and we have .·.

$$P = \frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''} + \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$= \frac{2\,[3] + (\alpha + \alpha^2)\,[12]}{[2] + (\alpha + \alpha^2)\,[1^2]}$$

$$Q = \frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''} \times \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$= \frac{[4] + (\alpha + \alpha^2)\,[2^2]}{[2] + (\alpha + \alpha^2)\,[1^2]}$$

If in these we put for the numerical expressions their values taken from the annexed tables, we then obtain, since $a + a^2 = -1$,

$$P = \frac{2A^3 - 7AB + 9C}{A^2 - 3B}$$

$$Q = \frac{4^4 - 4A^2B + B^2 + 6AC}{A^2 - 3B}$$

If we substitute these values of P and Q in the equation $a^2 - Pa + Q = 0$, we then obtain by its solution the two values of a, expressed by the coefficients A, B, C of the given equation.

Having, however, found the two values of a, we then obtain from them the two values of b, by means of the equation (7).

$$b = -\frac{[2] + a[1]}{3}$$

SECTION CXXXI.

Prob. To reduce the given equation of the fourth degree $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ to an equation of the form $y^4 - Vy^2 + Z = 0$, we must make use of the auxiliary equation $x^2 + ax + b = y$: required to determine, a priori, the degrees of the equations, on which the coefficients a, b, depend.

Solution 1. Since in the transformed equation there are only even powers of y, then two and two are equal and opposite each other. If $\therefore$ we denote these by y', $-y'$, y'', $-y''$, and the roots of the given equation by x', x'', x''', x^{IV}, we then have, when these values of x and y are made use of in the auxiliary equation,

$$\begin{aligned} x'^2 + ax' + b &= y' \\ x''^2 + ax'' + b &= -y' \\ x'''^2 + ax''' + b &= y'' \\ x'^{v2} + ax'^{v} + b &= -y'' \end{aligned}$$

If we add the two first and the two last of these equations together, we obtain

$$\begin{aligned} x'^2 + x''^2 + a(x' + x'') + 2b &= 0 \\ x'''^2 + x'^{v2} + a(x''' + x'^{v}) + 2b &= 0 \end{aligned}$$

and hence

$$a = -\frac{x'^2 + x''^2 - x'''^2 - x'^{v2}}{x' + x'' - x''' - x'^{v}}$$

2. Since it matters not, which values of x and y are considered as belonging to each other, consequently a has not only this single value, but all the values at once, which can arise from all the possible transpositions of the roots. But the function which was found for a, belongs, as may be easily seen, to that class of functions, for which

$$f\colon (x')\,(x'')\,(x''')\,(x'^{v}) = f\colon (x'')\,(x')\,(x''')\,(x'^{v}) = f\colon (x')\,(x'')\,(x'^{v})\,(x''') = f\colon (x''')\,(x'^{v})\,(x')\,(x'')$$

consequently to the same class, which we have treated of in the second example, § LV. This function .·., as was there found, has no more than three different values, which are given by the types $f\colon (x')\,(x'')\,(x''')\,(x'^{v})$, $f\colon (x''')\,(x')\,(x'^{v})\,(x'')$, $f\colon (x'')\,(x''')\,(x'^{v})\,(x')$. If for these symbols we put what is denoted by them, we then obtain the following three values for a:

$$-\frac{x'^2 + x''^2 - x'''^2 - x'^{v2}}{x' + x'' - x''' - x'^{v}}$$

$$-\frac{x'''^2 + x'^2 - x'^{v2} - x''^2}{x''' + x' - x'^v - x''}$$

$$-\frac{x''^2 + x'''^2 - x'^{v2} - x'^2}{x'' + x''' - x'^v - x'}$$

and consequently the coefficient a depends on an equation of the third degree.

3. Further, if we add the four equations in 1, we obtain

$x'^2 + x''^2 + x'''^2 + x'^{v2} + a(x' + x'' + x''' + x'^v) + 4b = o$

or

$$[2] + a\ [1] + 4b = o$$

∴

$$b = -\frac{[2] + a\,[1]}{4}$$

Now, since a has three values, b has also three, and consequently b depends on an equation of the third degree.

Remark. Hence, if it were required, we could find the equations for a and b, by the method sufficiently well known from the third chapter; but since this subject possesses no difficulty, I shall not detain my readers any longer with it.

SECTION CXXXII.

PROB. To reduce the equation of the fourth degree $x^4 - Ax^3 + Bx^2 - Cx + D = o$ to one of two terms, $y^4 - V = o$, we must assume an auxiliary equation with three unknown magnitudes, because three terms must vanish. Let $x^3 + ax^2 + bx + c = y$ be this auxiliary

equation. Required to find the degree of the equations, on which the coefficients a, b, c depend.

Solution 1. The roots of the equation $y^4 - V = o$ are, when we put $\sqrt[4]{V} = y'$, y', $-y'$, $+ y'\sqrt{-1} - y'\sqrt{-1}$. The roots of the given equation x', x'', x''', x'^{v}, may correspond to these values of y in $1 . 2 . 3 . 4 = 24$ different ways; viz. as often as these values can be transposed. For while these are undetermined, it is quite immaterial, which values of x and y are considered as belonging to one another. We can, however, as was also done in the two preceding sections, combine the values of x and y together in any way we please, and then introduce in the results the above-mentioned transpositions.

2. The substitution of the values of x and y in the auxiliary equation gives

$$(\phi)\quad \begin{aligned} x'^3 + ax'^2 + bx' + c &= y' \\ x''^3 + ax''^2 + bx'' + c &= -y' \\ x'''^3 + ax'''^2 + bx''' + c &= y'\sqrt{-1} \\ x'^{v3} + ax'^{v2} + bx'^{v} + c &= -y'\sqrt{-1} \end{aligned}$$

3. If the two first and the two last equations be added together, we obtain

$$x'^3 + x''^3 + a(x'^2 + x''^2) + b(x' + x'') + 20 = o$$
$$x'''^3 + x'^{v3} + a(x'''^2 + x'^{v2}) + b(x''' + x'^{v}) + 20 = o$$

and when these again are subtracted from one another

$$x'^3 + x''^3 - x'''^3 - x'^{v3} + a(x'^2 + x''^2\, x'''^2 - x'^{v2})$$
$$+ b(x' + x'' - x''' - x'^{v}) = o.$$

4. But if we subtract the two first and the two last of the equations (ϕ) from one another, we obtain

$$x'^3 - x''^3 + a\,(x'^2 - x''^2) + b\,(x' - x'') = 2y'$$
$$x'''^3 - x'^{v3} + a\,(x'''^2 - x'^{v2}) + b\,(x''' - x'^{v}) = 2y'\sqrt{-1}$$

and if we multiply the second of these equations by $\sqrt{-1}$, and then add it to the first,

$$\begin{aligned} & x'^3 - x''^3 + (x'''^3 - x'^{v3})\sqrt{-1} \\ & + a\,[x'^2 - x''^2 + (x'''^2 - x'^{v2})\sqrt{-1}] \\ & + b\,[x' - x'' + (x''' - x'^{v})\sqrt{-1}] = 0 \end{aligned}$$

5. Since this equation, as also the one found in 3, only contains a and b, ∴ they can conjointly serve to determine these two coefficients. Thus if we eliminate b, and, for the sake of brevity, put

$$\left(\begin{matrix} (x'^3 + x''^3 - x'''^3 - x'^{v3})\,(x' - x'') \\ -(x'^3 - x''^3)\,(x' + x'' - x''' - x'^{v}) \end{matrix}\right) = M$$

$$\left(\begin{matrix} (x'^3 + x''^3 - x'''^3 - x'^{v3})(x''' - x'^{v}) \\ -(x'''^3 - x'^{v3})\,(x' + x'' - x''' - x'^{v}) \end{matrix}\right) = N$$

$$\left(\begin{matrix} (x'^2 + x''^2 - x'''^2 - x'^{v2})\,(x' - x'') \\ -(x'^2 - x''^2)\,(x' + x'' - x''' - x'^{v}) \end{matrix}\right) = P$$

$$\left(\begin{matrix} (x'^2 + x''^2 - x'''^2 - x'^{v2})\,(x''' - x'^{v}) \\ -(x'''^2 - x'^{v2})\,(x' + x'' - x''' - x'^{v}) \end{matrix}\right) = Q$$

we then obtain the equation

$$M + N\sqrt{-1} + (P + Q\sqrt{-1})\,a = 0$$

and hence

$$a = -\,\frac{M + N\sqrt{-1}}{P + Q\sqrt{-1}}$$

6. If in the functions denoted by M, N, P, Q, we substitute x''' for x', x'^{v} for x'', x'' for x''', and x' for

x^{IV}, then M is transformed into $-N$, N into M, P into $-Q$, and Q into P. Therefore the expression found for a, by this substitution, is transformed into $-\frac{-N+M\sqrt{-1}}{-Q+P\sqrt{-1}}$, or, if we multiply the numerator and denominator by $-\sqrt{-1}$, into $-\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}}$, consequently the same as before. The expression for a belongs ∴ to that class of functions for which

$$f\colon (x')\,(x'')\,(x''')\,(x^{IV}) = f\colon (x''')\,(x^{IV})\,(x'')\,(x').$$

7. From this equation we obtain (§ LIV) the following period of equal types:

$$\begin{array}{l} f\colon (x')\,(x'')\,(x''')\,(x^{IV}) \\ f\colon (x''')\,(x^{IV})\,(x'')\,(x') \\ f\colon (x'')\,(x')\,(x^{IV})\,(x''') \\ f\colon (x^{IV})\,(x''')\,(x')\,(x'') \end{array}$$

Amongst the $1.2.3.4=24$ values, which the coefficient a contains by the transposition of the roots x', x'', x''', x^{IV}, there are 6 times 4 equal values, consequently only 6 different ones, which are denoted by the following types:

$$\begin{array}{l} f\colon (x')\,(x'')\,(x''')\,(x^{IV}) \\ f\colon (x')\,(x'')\,(x^{IV})\,(x''') \\ f\colon (x')\,(x''')\,(x'')\,(x^{IV}) \\ f\colon (x')\,(x''')\,(x^{IV})\,(x'') \\ f\colon (x')\,(x^{IV})\,(x'')\,(x''') \\ f\colon (x')\,(x^{IV})\,(x''')\,(x'') \end{array}$$

and which are obtained merely by the transposition of the three roots x'', x''', x^{IV}. Since ∴ a has six different

values, these magnitudes consequently depend on an equation of the sixth degree.

8. In the same way it may be proved, that b likewise depends on an equation of the sixth degree. For if we eliminate a instead of b in the equations found in 3 and 4, we obtain, when for the sake of brevity we put

$$\left(\begin{array}{l} (x'^3+x''^3-x'''^3-x'^{V3})\,(x'^2-x''^2) \\ -(x'^3-x''^3)\,(x'^2+x''^2-x'''^2-x'^{V2}) \end{array}\right) = M$$

$$\left(\begin{array}{l} (x'^3+x''^3-x'''^3-x'^{V3})\,(x'''^2-x'^{V2}) \\ -(x'''^3-x'^{V3})\,(x'^2+x''^2-x'''^2-x'^{V2}) \end{array}\right) = N$$

$$\left(\begin{array}{l} (x'+x''-x'''-x'^{V})\,(x'^2-x''^2) \\ -(x'-x'')\,(x'^2+x''^2-x'''^2-x'^{V2}) \end{array}\right) = P$$

$$\left(\begin{array}{l} (x'+x''-x'''+x'^{V})\,(x'''^2-x'^{V2}) \\ -(x'''-x'^{V})\,(x'^2+x''^2-x'''^2-x'^{V2} \end{array}\right) = Q$$

the equation

$$M+N\sqrt{-1}+b\,(P+Q\sqrt{-1})=0$$

and hence

$$b=-\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}}$$

If we substitute the roots x', x'', x''', x'^{V}, for the roots x''', x'^{V}, x'', x', respectively, then the expressions M, N, P, Q, are respectively transformed into $-N$, M, $-Q$, P; and $\therefore$ it may be shewn, as in 6, that the expression found for b is of the form $f:\ (x')\,(x'')\,(x''')\,(x'^{V}) = f:\ (x''')\,(x'^{V})\,(x'')\,(x')$, and consequently has no more than six different values. Consequently b also depends on an equation of the sixth degree.

9. If the four equations in 2 be added together, we obtain

$$[3] + a[2] + b[1] + 4c = 0$$

$\therefore$

$$c = -\frac{[3] + a[2] + b[1]}{4}$$

consequently if the coefficients a and b are found, we then also have c.

10. Therefore the magnitudes a, b, depend on equations of the sixth degree, and consequently the transformation required in the Problem is not practicable, when these equations are not reducible to equations of the second or third degree. But of the possibility or impossibility of such a reduction, we can convince ourselves syllogistically without completing the calculation, as the following § will show. Moreover, I must again remind my readers, that the magnitudes a and b, are homogeneous functions of the roots x', x'', x''', $x^{\prime v}$, (§ L); and it will be seen from the following chapter, that in this case it is quite sufficient, merely to have found one of these two magnitudes, because then the other may always be derived directly from it, without its being first necessary to solve a new equation. I shall $\therefore$ merely confine myself to inquiries respecting the equation for a.

SECTION CXXXIII.

PROB. Required to find, whether the equation of the sixth degree, on which the magnitude a of the preced-

ing § depends, is reducible to equations of a lower degree.

Solution 1. In the preceding § we have seen that the six roots of the equation for a, are expressed by the following types:

$$f:\ (x')\,(x'')\,(x''')\,(x^{\prime v}),\quad f:\ (x')\,(x'')\,(x^{\prime v})\,(x''')$$
$$f:\ (x')\,(x''')\,(x'')\,(x^{\prime v}),\quad f:\ (x')\,(x''')\,(x^{\prime v})\,(x'')$$
$$f:\ (x')\,(x^{\prime v})\,(x'')\,(x'''),\quad f:\ (x')\,(x^{\prime v})\,(x''')\,(x'').$$

I shall now assume, that the values of a, which correspond to the two first types, are the roots of an equation of the second degree,

$$a^2 - pa + q = o$$

then, according to the nature of equations,

$$p = f:\ (x')\,(x'')\,(x''')\,(x^{\prime v}) + f:\ (x')\,(x'')\,(x^{\prime v})\,(x''')$$
$$q = f:\ (x')\,(x'')\,(x''')\,(x^{\prime v}) \times f:\ (x')\,(x'')\,(x^{\prime v})\,(x''')$$

2. However, from this compound form, it immediately follows that p and q are functions of the form

$$\phi:\ (x')\,(x'')\,(x''')\,(x^{\prime v}) = \phi:\ (x')\,(x'')\,(x^{\prime v})\,(x''')$$

because in the transformation of the roots x''', $x^{\prime v}$, $f:\ (x')\,(x'')\,(x''')\,(x^{\prime v})$ into $f:\ (x')\,(x'')\,(x^{\prime v})\,(x''')$, and conversely, when the last is transformed into the first, no change arises in the expressions for p and q.

3. But by the preceding § the function denoted by f is of the form

$$f:\ (x')\,(x'')\,(x''')\,(x^{\prime v}) = f:\ (x''')\,(x^{\prime v})\,(x'')\,(x')$$

consequently also p and q are functions of this form. Therefore p and q are functions of the form

$$\phi : (x') (x'') (x''') (x^{IV}) = \phi : (x''') (x^{IV}) (x'') (x')$$
$$= \phi : (x') (x'') (x^{IV}) (x''')$$

and this form gives no more than three different values, viz.

$$\phi : (x') (x'') (x''') (x^{IV})$$
$$\phi : (x') (x''') (x'') (x^{IV})$$
$$\phi : (x') (x^{IV}) (x'') (x''')$$

4. Therefore the coefficients p, q depend only on two cubic equations

$$p^3 - A'p^2 + B'p + C' = 0$$
$$q^3 - A''q^2 + B''q - C'' = 0$$

If, therefore, we denote the three roots of the first equation by p', p'', p''', and the three roots of the second by q', q'', q''', we then obtain the three following equations of the second degree:

$$a^2 - p'a + q' = 0$$
$$a^2 - p''a + q'' = 0$$
$$a^2 - p'''a + q''' = 0$$

into which ∴ the equation of the sixth degree for a may be analyzed.

Corollary. If we substitute the roots x''', x^{IV}, for one another in the expressions M, N, P, Q, in 5 of the preceding §, the expressions M and P remain the same; the expressions N and Q, on the contrary, are transformed into $-N$, $-Q$. If ∴ we put

$$f : (x') (x'') (x''') (x^{IV}) = -\frac{M + N\sqrt{-1}}{P + Q\sqrt{-1}}$$

then

$$f:\ (x')\ (x'')\ (x^{\prime V})\ (x''') = -\frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

Therefore

$$p = -\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}} - \frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

$$q = -\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}} \times -\frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

or

$$p = -\frac{2(MP+NQ)}{P^2+Q^2}, \quad q = \frac{M^2+N^2}{P^2+Q^2}$$

From these functions the equations for p and q may be actually found by the method given in the third chapter.

SECTION CXXXIV.

The results in the two foregoing Sections may also be immediately derived from considering the equations (ϕ) in 2, § CXXXII. Thus, in § CXXXII, in order to combine the four values of x with the four values of y in all possible ways, instead of pre-supposing a transformation of the former, as was there done, we assume a transformation of the latter. The equations (ϕ), by the first as well as the other transformation, undergo twenty-four changes; and since each such change gives a value of a, we obtain twenty-four values of a, of which, as we have already seen, no more than six are different.

This conclusion, however, might have been foreseen, à priori, without knowing the value of a. Amongst the twenty-four combinations of the roots x', x'', x''', $x^{\prime V}$ with y', $-y'$, $y'\sqrt{-1}$, $-y'\sqrt{-1}$, there are also the following

$$x'^3 + ax'^2 + bx' + c = \ \ y'\sqrt{-1}$$
$$x''^3 + ax''^2 + bx'' + c = -y'\sqrt{-1}$$
$$x'''^3 + ax'''^2 + bx''' + c = -y'$$
$$x'^{v3} + ax'^{v2} + bx'^{v} + c = \ \ y'$$

and these four equations might have been obtained from those in 2, § CXXXII, by substituting throughout $y'\sqrt{-1}$ for y'. However, by such a substitution as this, the value of a can undergo no change; for after we have eliminated y, it matters not in the least what we substitute for it. Hence it follows, that these equations must give the same value of a, as the former; and since the former might also have been obtained from the latter by substituting the roots x', x'', x''', x'^{v}, for x''', x'^{v}, x'', x', consequently it follows, that the expression for a must be such, that it suffers no change by the above substitution; it must ∴ necessarily be of the form

$$f:\ (x')\ (x'')\ (x''')\ (x'^{v}) = f:\ (x''')\ (x'^{v})\ (x'')\ (x')$$

which coincides with 6, § CXXXII.

If in the equations (ϕ) in 2, § CXXXII, we substitute $-\sqrt{-1}$, for $\sqrt{-1}$, we again obtain a new set of equations, which only differ from the equations (ϕ) in this, that in the former x'^{v} is combined with $y\sqrt{-1}$, and x''' with $-y'\sqrt{-1}$; whereas in the latter the reverse of this is the case; we might ∴ have obtained them also merely by substituting x''' for x'^{v}. But hence it follows, that the expression for a must be such, that we obtain $f:\ (x')\ (x'')\ (x'^{v})\ (x''')$ from $f:\ (x')\ (x'')\ (x''')\ (x'^{v})$ when we merely put $-\sqrt{-1}$ for $\sqrt{-1}$, which agrees with § CXXXIII, Corollary.

Further, since the functions p, q of the preceding §,

as sum and product of the two functions $f: (x') (x'') (x''') (x^{IV})$, $f: (x') (x'') (x^{IV}) (x''')$, by the substitution of $-\sqrt{-1}$ for $\sqrt{-1}$ suffer no change, so likewise they undergo no change by the substitution of x''' for x' ; and they are consequently functions of the form

$$\phi: (x') (x'') (x''') (x^{IV}) = \phi: (x') (x'') (x^{IV}) (x''')$$

and since they are also functions of the form

$$\phi: (x') (x'') (x''') (x^{IV}) = \phi: (x''') (x^{IV}) (x'') (x')$$

because they are compounded of these ; it follows then, as in the foregoing §, that they can have three different values only, and they consequently depend on equations of the third degree.

SECTION CXXXV.

Prob. To transform the equation of the fifth degree

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

into one of two terms $y^5 - V = 0$, assume the auxiliary equation:

$$x^4 + ax^3 + bx^2 + cx + d = y$$

required to find the degree of the equations, which must be solved, in order to determine the coefficients a, b, c, d.

Solution 1. If α denote one of the imaginary roots of the equation $y^5 - 1 = 0$, and we put $\sqrt[5]{V} = y'$, then y', $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, $\alpha^4 y'$, as we know already, are the five roots of the equation $y^5 - V = 0$. If we introduce these values of y, together with the values of x, into the

auxiliary equation, we then obtain the five following equations.

$$\begin{aligned}
x'^4 + ax'^3 + bx'^2 + cx' + d &= y' \\
x''^4 + ax''^3 + bx''^2 + cx'' + d &= \alpha y' \\
x'''^4 + ax'''^3 + bx'''^2 + cx''' + d &= \alpha^2 y' \\
x'^{v4} + ax'^{v3} + bx'^{v2} + cx'^{v} + d &= \alpha^3 y' \\
x^{v4} + ax^{v3} + bx^{v2} + cx^{v} + d &= \alpha^4 y'
\end{aligned}$$

2. By these equations, which, with reference to a, b, c, d, are only of the first degree, we can express each of these coefficients by the roots x', x'', x''', x'^{v}, x^{v}, and then, as was done in the foregoing § in equations of the third and fourth degrees, permute in the expressions thus found the above roots as often as possible; in the present instance $1 . 2 . 3 . 4 . 5 = 120$ times; the number of different values, which we obtain by these means, will then determine the degree of the equations on which the magnitudes a, b, c, d depend. But let it not be supposed, that the elimination, and also the comparison of the 120 results, are very laborious in themselves: another particular inquiry will always be required, if the equations obtained last cannot be reduced lower. We wish ∴ to try whether, from the nature of the above equations themselves, there are not certain indications by which we may more easily attain the desired object.

3. Since the number of the values of each of the coefficients a, b, c, d, say a, (the same conclusions may be made for the others) has its foundation merely in this, that the values of x and y may be combined in more ways

than one, it only remains to examine the results of these different combinations. But we obtain all the possible combinations of the values of x and y, when in the above five equations we either permute the values of x, or the values of y in every way. If we make choice of the first method, and, according to Hindenburg's Rule of Permutation, let x' retain its place in the first equation, when we transpose the roots x'', x''', x^{IV}, x^{V}, we then obtain twenty-four sets, each consisting of five equations. If after this we introduce the roots x'', x''', x^{IV}, x^{V}, successively into the first equation, when we permute the four remaining roots, we then obtain 120 sets in all, each consisting of five equations, and consequently all the combinations of the values of x with the values of y. Each such set gives a value of a; $\therefore$ collectively 120 values of a.

4. Now I assert, that amongst these 120 values of a, there are no more than twenty-four different ones, and that these different values are obtained from the first twenty-four sets. For, in the first place, it is easily seen, that it is quite immaterial whether we introduce the roots x'', x''', x^{IV}, x^{V}, successively in the first equation, and permute the other roots, or whether we substitute in all the five equations $\alpha y'$, $\alpha^2 y'$ $\alpha^3 y'$, $\alpha^4 y'$, successively for y', and after each such substitution permute the roots x'', x''', x^{IV}, x^{V}, when x' retains its place in the first equation. Now, since y' occurs only once in the value of a, because it was eliminated at the very beginning in the above equations, so with respect to this value it matters not what we substitute for y'; and $\therefore$, by the above-men-

tioned substitutions, we shall find no other values than those which we obtain, when we let x' retain its place, and merely permute the roots x'', x''', x^{IV}, x^{V}.

5. Now, we are certain that the expression for a, which we obtain by actual calculation, is such, that amongst the 120 values which arise from the transposition of all the five roots, x', x'', x''', x^{IV}, x^{V}, there are no more than twenty-four different ones, and that these last values are those which are obtained exclusively from the transposition of the four roots x'', x''', x^{IV}, x^{V}. The equation for a ∴ rises no higher than the twenty-fourth degree. We shall now see whether this equation cannot be reduced to others of lower degrees.

6. Since we do not know the expression for a, we shall ∴ immediately assume the transposition of the roots x'', x''', x^{IV}, x^{V}, in the above five equations, while in this we set the first completely aside, because x' may be considered as a constant magnitude. For this purpose, according to Hindenburg's Rule of Transposition, we again let x'' retain its place, and only permute the roots x''', x^{IV}, x^{V}, we then obtain six sets, each consisting of five equations. If, after this, we put x''', x^{IV}, x^{V}, successively for x'' in the second equation, and after every such substitution permute the other three roots, we then obtain in all the twenty-four sets of equations, which give the twenty-four different values of a.

7. Instead of the above method, we can also make use of the following one. First put in the above five equa-

tions α^2, α^3, α^4 successively for α, we then obtain, since $\alpha^6=\alpha$, $\alpha^7=\alpha^2$, $\alpha^8=\alpha^3$, &c., the four following combinations of the values of x and y:

$$\begin{array}{ccccc}
x', & x'', & x''', & x^{IV}, & x^{V} \\
\hline
y', & \alpha y', & \alpha^2 y', & \alpha^3 y', & \alpha^4 y' \\
y', & \alpha^2 y', & \alpha^4 y', & \alpha y', & \alpha^3 y' \\
y', & \alpha^3 y', & \alpha y', & \alpha^4 y', & \alpha^2 y' \\
y', & \alpha^4 y', & \alpha^3 y', & \alpha^2 y', & \alpha y'
\end{array}$$

and if we permute in each of these combinations the three roots x''', x^{IV}, x^{V}, we then obtain the twenty-four combinations of the values of x and y, which are possible only under the condition that x' continues to be combined with y'.

8. If ∴ we had expressed the coefficient a, from the five equations in 1, by the roots x', x'', x''', x^{IV}, x^{V}, it would only have been necessary, in order to find its twenty-four different values, to have transformed α into α^2, α^3, α^4, in each of these four values merely to have permuted the three roots x''', x^{IV}, x^{V}.

9. Now, if we assume that the values of a, which give the four combinations in 7, are the roots of an equation of the fourth degree

$$a^4 - pa^3 + qa^2 - ra + s = 0$$

and if we denote these roots by a', a'', a''', a^{IV}, then

$$p = a' + a'' + a''' + a^{IV}$$
$$q = a'a'' + a'a''' + a'a^{IV} + a''a''' + \&c.$$
&c.

Now, since the functions a', a'', a''', a^{IV}, are such, that by the substitution of α for α^2, α^3, α^4, they merely are transformed into one another, consequently p, q, r, s, are symmetrical functions of the roots of the equation $y^5-1=0$, and $\therefore$ they cannot contain α. The functions p, q, r, s, consequently contain neither y' nor α; and $\therefore$ in all the transpositions of the roots x', x'', x''', x^{IV}, x^{V}, have no more than six different values, viz. only those which arise from the permutation of the roots x''', x^{IV}, x^{V}.

10. Consequently the coefficients p, q, r, s, all depend on equations of the sixth degree; and if we do not mind the trouble, these equations may be actually found by the method given in the third chapter. But whether they are capable of any further reduction or not, will be seen in the sequel. I shall now only observe, that it is quite sufficient to solve one of these equations, viz, that for p, because then, as will be seen in the following chapter, the coefficients q, r, s, may be found directly.

SECTION CXXXVI.

PROB. To transform the given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

into one of two terms, viz. $y^n - V = 0$, we assume the auxiliary equation

$$x^{n-1} + ax^{n-1} + bx^{n-2} + \ldots\ldots + kx + l = y$$

with $n-1$ undetermined coefficients $a, b, c, \ldots\ldots k, l$: on the supposition that n is a prime number, required to determine the degrees of the equations, which must be solved, in order to find the assumed coefficients.

Solution 1. If we put $\sqrt[n]{V} = y'$, and α denote an imaginary root of the equation $y^n - 1 = 0$, then y', $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$, are the n roots of the equation $y^n - V = 0$. If we combine these values with the roots of the given equation, we obtain the following n equations:

$$x'^{n-1} + ax'^{n-2} + bx'^{n-3} + \ldots\ldots + kx' + l = y'$$
$$x''^{n-1} + ax''^{n-2} + bx''^{n-3} + \ldots\ldots + kx'' + l = \alpha y'$$
$$x'''^{n-1} + ax'''^{n-2} + bx'''^{n-3} + \ldots\ldots + kx''' + l = \alpha^2 y'$$
&c.

from which we first eliminate y', and from the $n - 1$ equations obtained by this elimination, the values a, b, c, d, &c. may be determined by the roots x', x'', x''', x^{iv}, &c.

2. The coefficient a (and this, as well as what follows, obtains also of b, c, d, &c.) in general contains as many values as the n values of x may be combined with the n values of y, in sets of n equations, as those in 1, on the condition, that every such set is different from the others in the same set. Of these combinations, however, there are exactly as many as there are transpositions in the n roots in the above set of n equations; consequently the number of the values which the coefficient a can have $= 1 . 2 . 3 \ldots\ldots n$. Consequently also the equation on which a depends, must be of the $1 . 2 . 3 \ldots\ldots n$th degree, supposing that amongst these values there are no equal ones.

3. When in the above equations we substitute $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$ successively for y', we then

obtain n sets of equations, in which the values of x and y are arranged according to the following scheme:

$$\begin{array}{cccccc}
x', & x'', & x''', & x^{IV}, & x^{V}, & \ldots\ldots\ldots\; x^{(n)} \\
\hline
y', & \alpha y', & \alpha^2 y', & \alpha^3 y', & \alpha^4 y', & \ldots\ldots\ldots\; \alpha^{n-1} y' \\
\alpha y', & \alpha^2 y', & \alpha^3 y', & \alpha^4 y', & \alpha^5 y', & \ldots\ldots\ldots\; y' \\
\alpha^2 y', & \alpha^3 y', & \alpha^4 y', & \alpha^5 y', & \alpha^6 y', & \ldots\ldots\ldots\; \alpha y' \\
\alpha^3 y', & \alpha^4 y', & \alpha^5 y', & \alpha^6 y', & \alpha^7 y', & \ldots\ldots\ldots\; \alpha^2 y' \\
\cdot\;\;\cdot & \cdot\;\;\cdot & \cdot\;\;\cdot & \cdot\;\;\cdot & \cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot \\
\alpha^{n-2} y', & \alpha^{n-1} y', & y', & \alpha y', & \alpha^2 y', & \ldots\ldots\ldots\; \alpha^{n-3} y' \\
\alpha^{n-1} y', & y', & \alpha y', & \alpha^2 y', & \alpha^3 y', & \ldots\ldots\ldots\; \alpha^{n-2} y'.
\end{array}$$

Here we find, as may be seen on inspection, x' always united with another value of y, while at the same time the remaining $n-1$ values of x are combined with the remaining $n-1$ values of y. If now, in each of these n combinations we permute the roots x'', x''', x^{IV} ...$x^{(n)}$, we then always obtain $1 \,.\, 2 \,.\, 3$ ...... $n-1$ combinations, and consequently from the whole together all the $1 \,.\, 2 \,.\, 3$ ...... n combinations, in which the values of x can enter with the values of y.

4. In order ∴ to find the results of all the possible combinations of the values of x and y, it is only necessary, in the above equations, to substitute $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$,...$\alpha^{n-1} y'$ successively for y', then transpose the roots x'', x''', x^{IV}, ... $x^{(n)}$ in all possible ways, and from each set thus obtained find a value of a; or, which is here the same, find first the expression for a, and then make use of the above substitution and transposition. But since y' has totally vanished in the expression for a (1), so by the substitution of $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$ for y', this expression

suffers no change, and $\therefore$ there are no more different values of this magnitude, than arise from the transposition of the $n-1$ roots x'', x''', $x^{\prime v}$, ... $x^{(n)}$.

5. Consequently the magnitude a depends on an equation of the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$th degree, and its roots are the values of the expression for this magnitude, which arises from the transposition of the roots x'', x''', ... $x^{(n)}$. Hence, then, this equation may actually be found, by the method given in the third chapter, and may be expressed in known magnitudes.

6. Since the unequal values of a belong exclusively to the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ sets, which arise from the transposition of the $n-1$ roots x'', x''', $x^{\prime v}$, ... $x^{(n)}$, in the above equations, it is allowable to consider the first equation, together with its roots x', y', as constant and invariable, and consequently we only require to take into consideration the $n-1$ equations

$$x''^{n-1} + ax''^{n-2} + \ldots\ldots\ldots + kx'' + l = \alpha y'$$
$$x'''^{n-1} + ax'''^{n-2} + \ldots\ldots\ldots + kx''' + l = \alpha^2 y'$$
&c.

If in these equations we transpose the roots x'', x''', ...$x^{(n)}$, in all possible ways, we obtain all the combinations between these roots and the roots $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$. But these combinations may also be found as follows.

7. In § LXXXVII, Cor., it was shown that, when in the series of roots α, α^2, α^3, ... α^{n-1} we successively substi-

tute α^2, α^3, α^4, ... α^{n-1} for α, the same roots, but in a different order, always present themselves. Hence it follows, that when in the equations in 6, we substitute successively α^2, α^3, α^4, ... α^{n-1} for α, the $n - 1$ sets of equations thus obtained, only differ from one another in the combination of the values of x with the values of y. Now since also every set has also another of the magnitudes $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$ in the first place, it follows, that we obtain all the combinations of the values of x with the values of y, when in each set we permute the $n - 2$ roots x''', x^{IV}, ... $x^{(n)}$ in the $n - 2$ last equations in all possible ways.

8. The changes which we have effected with the equations themselves, we can also effect with their result, the expression for a. Thus, if by the equations in 1 we have expressed the magnitude a by x', x'', x''',...$x^{(n)}$, we then substitute in it successively α^2, α^3, α^4,......α^{n-1} for α, and permute in each of the values thus obtained, merely the roots x''', x^{IV}, x^{V},......$x^{(n)}$, while we let x' and x'' retain their places.

9. Now, if we denote the $n-1$ values of a, which arise from the substitution of α^2, α^3, α^4, ... α^{n-1}, for α, by a', a'', a''',......$a^{(n)}$, and assume that they are the roots of the following equation of the $n-1$th degree:

$$a^{n-1} - pa^{n-2} + qa^{n-3} - ra^{n-4} + \&c. = 0$$

then the coefficients p, q, r, &c. merely as functions of a', a'', a''', ... $a^{(n-1)}$, in like manner can have no more different values than the $1 \,.\, 2 \,.\, 3......n-1$, which

arise from the substitution of α for α^2, α^3, α^4,......α^{n-1}, and from the transposition of the roots x''', x^{IV}, x^{V},...$x^{(n)}$. But since these coefficients are also at the same time symmetrical functions of α, α^2, α^3,......$\alpha^{(n-1)}$, consequently of the roots of the equation $y^n - 1 = 0$; $\therefore$, by the foregoing chapter, they must be rational, and consequently do not contain α. Wherefore these coefficients have no more than the $1 \,.\, 2 \,.\, 3 \ldots n-2$ different values, which arise from the transposition of the roots x''', x^{IV}, x^{V}, ...$x^{(n)}$, and $\therefore$ they all depend on equations of the $1 \,.\, 2 \,.\, 3 \ldots\ldots n-2$th degree.

10. Let $1 \,.\, 2 \,.\, 3 \ldots\ldots n-1 = \mu$, and let

$$p^{\mu} + A'p^{\mu-1} + B'p^{\mu-2} + C'p^{\mu-3} + \&c. = 0$$

be the equation for p; then, as is already known from the third chapter, the coefficients A', B', C', &c. may always be found, and expressed rationally by the coefficients A, B, C, &c. of the given equation $x^n + Ax^{n-1} + \&c. = 0$. If, then, we can solve this equation, and from it determine the values of p, then also we may, as will be shown in the following chapter, find the coefficients q, r, s, &c. directly, and without solving any other equation. Now, if we denote the values of p, q, r, &c. which we thus obtain, by p', q', r', &c., p'', q'', r'', &c, p''', q''', r''', &c. &c. we then obtain the following $1 \,.\, 2 \,.\, 3 \ldots n-2$ equations:

$$\begin{aligned}
a^{n-1} + p'a^{n-2} + q'a^{n-3} + r'a^{n-4} + \&c. &= 0 \\
a^{n-1} + p''a^{n-2} + q''a^{n-3} + r''a^{n-4} + \&c. &= 0 \\
a^{n-1} + p'''a^{n-2} + q'''a^{n-3} + r'''a^{n-4} + \&c. &= 0 \\
\&c.
\end{aligned}$$

into which the equation for a in 5 may be analyzed. But

whether or not the equation for p is capable of any reduction, this is not yet the proper place to inquire.

Remark. Hence it follows, that an equation of the nth degree, when n is a prime number, leads, according to this mode of transformation, to an equation of the $1 . 2 . 3 ... n-2$th degree; consequently an equation of the fifth degree to an equation of the sixth degree, and an equation of the seventh degree leads even to one of the 120th degree; and so on.

SECTION CXXXVII.

Prob. All that has been said in the problem in the foregoing § obtains, only n is a compound number: determine the degree of the equations on which the assumed coefficients a, b, c, d, &c. depend.

Solution 1. When we suppose α, not to be, as in the preceding §, any arbitrary imaginary root of the equation $y^n - 1 = 0$, but only a primitive root of it, then indeed all the conclusions made in 1, 2, 3, 4, 5, 6, of the preceding § are applicable to this case; on the other hand, the following solutions, (7, 8, 9, 10), on account of this very circumstance, must undergo some alterations. Thus, if we substitute here, as in 7 of the preceding §, in the series of roots $\alpha, \alpha^2, \alpha^3, \ldots\ldots \alpha^{n-1}$ the powers $\alpha^2, \alpha^3, \alpha^4, \ldots\ldots \alpha^{n-1}$ indiscriminately for α, we shall not always find again the same roots, but this will only be the case for those powers amongst them, $\alpha^\nu, \alpha^\pi, \alpha^\rho$, &c. whose exponents ν, π, ρ, &c. have no common measure with n,

because these only are primitive roots of the equation $y^n-1=o$ (§ XCI). Thus, if α be a primitive root of the equation $y^6-1=o$, when in α, α^2, α^3, α^4, α^5, we successively substitute α^2, α^3, α^4, α^5, for α, we obtain the following results: α^2, α^4, 1, α^2, α^4; α^3, 1, α^3, 1, α^3; α^4, α^2, 1, α^4, α^2; α^5, α^4, α^3, α^2, α, of which only the last contains again the same roots.

2. Hence it follows, that what has been said in the preceding § in 7, and the following solutions, respecting the substitution of the roots α^2, α^3, α^4,......α^{n-1} for α, must be limited to α^ν, α^π, α^ϱ, &c. whose exponents ν, π, ρ, &c. have no common measure with n. If ∴ we assume, that λ is the number of the primitives α^ν, α^π, α^ϱ, &c. and that the λ values of a, which we obtain by the substitution of these roots for α, are the roots of the following equations:

$$a^\lambda + pa^{\lambda-1} + qa^{\lambda-2} + ra^{\lambda-3} + \&c. = o$$

consequently p (and the same obtains also of q, r, &c.) is such a function, as by the substitution of the root α for α^ν, α^π, α^ϱ, &c., or, which is the same, by the substitution of the root x'' for $x^{(\nu)}$, $x^{(\pi)}$, $x^{(\varrho)}$, &c. it remains unchanged. Now, since by these means all the $1 . 2 . 3 \ldots n-1$ values of p, taken λ and λ together, are equal, it follows, that this magnitude depends on an equation, whose degree

$$= \frac{1 . 2 . 3 \ldots\ldots n-1}{\lambda}$$

Moreover, in order actually to find the equation $a^\lambda + pa^{\lambda-1} + \&c. = o$, it is only required in the expression for a, which we obtain from 1 of the foregoing §, to eliminate

the root α, by means of that equation which only contains the primitive roots, the method to find which was given in § LXXXIX.

Remark. Therefore the equation of the nth degree, when n is a compound number, leads to an equation of the $\frac{1 \cdot 2 \cdot 3 \ldots\ldots n-1}{\lambda}$ th degree, in which λ denotes the number of the primitive roots of the equation $x^n - 1 = 0$; consequently an equation of the fourth degree leads to an equation of the third degree, because the equation $x^4 - 1 = 0$ has two primitive roots, α and α^3; an equation of the sixth degree to an equation of the sixtieth degree, because the equation $x^6 - 1 = 0$ has also two primitive roots, viz. α and α^5; an equation of the eighth degree to an equation of the 1260th degree, because the equation $x^8 - 1 = 0$ has four primitive roots, viz. α, α^3, α^5, α^7; and so on.

From this and the foregoing §, it follows, that the reduction of the equation $x^n + Ax^{n-1} + \&c. = 0$ to one of the form $y^n - V = 0$, always leads to a higher equation than the given one itself, whenever the given equation exceeds the fourth degree. However, I shall not enter here into the proof of other methods of transformations, because the whole of this subject will be considered hereafter in a higher point of view, to which this is only preparatory.

SECTION CXXXVIII.

When we make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$

$+ c\sqrt[n]{p^3} + \ldots\ldots + k\sqrt[n]{p^{n-1}}$, or more generally the equation $x = a + b\sqrt[n]{p} + c\sqrt[n]{p^2} + d\sqrt[n]{p^3} + \ldots\ldots + k\sqrt[n]{p^{n-1}}$ rational, we arrive, as we know from the foregoing chapter, at an equation of the nth degree, which I shall represent by $x^n + \mathfrak{A}x^{n-1} + \mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} +$ &c. $= 0$, in which the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. are certain rational functions of the magnitudes a, b, c, d, &c. and p. Conversely, if an equation of the nth degree $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} +$ &c. $= 0$ be given, and we assume that the roots have the above form, we then have, for the determination of a, b, c, d, &c. the n conditional equations $\mathfrak{A} = A$, $\mathfrak{B} = B$, $\mathfrak{C} = C$, &c. If we solve these equations, and from them determine a, b, c, &c. we then have at once the n roots of the given equation, when we substitute successively for $\sqrt[n]{p}$ its n values.

Consequently all depends on the solution of these conditional equations. How difficult and troublesome this solution must be for equations of rather high degrees, may be perceived from the form of those equations, which were found for the fifth degree in 7, § CIX. Waring and Euler thought, that in this way we must arrive at the general solution of equations, by properly arranging and finishing the calculation, without regard to the trouble. But this trouble may be spared by subjecting à priori (as M. Lagrange does in the third volume of the New Memoirs of the Berlin Academy) the method to a preliminary proof.

SECTION CXXXIX.

PROB. Let it be assumed that

$$x=a+b\sqrt[n]{p}+c\sqrt[n]{p^2}+d\sqrt[n]{p^3}+\ldots\ldots+k\sqrt[n]{p^{n-1}}$$

is a root of the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

on the supposition that n is a prime number, required to determine the degrees of the equations on which the coefficients a, b, c, d, &c. depend.

Solution 1. Put $\sqrt[n]{p} = y$; then y, αy, βy, γy, δy, &c. are the n values of $\sqrt[n]{p}$, when 1, α, β, γ, δ, &c. are the n roots of the equation $x^n - 1 = 0$. If $\therefore$ we denote the roots of the given equation by x', x'', x''', &c. we have the following n equations:

$$\begin{aligned}
x' &= a+by+cy^2+dy^3+\ldots\ldots+ky^{n-1}\\
x'' &= a+\alpha by+\alpha^2cy^2+\alpha^3dy^3+\ldots\ldots+\alpha^{n-1}ky^{n-1}\\
x''' &= a+\beta by+\beta^2cy^2+\beta^3dy^3+\ldots\ldots+\beta^{n-1}ky^{n-1}\\
x^{IV} &= a+\gamma by+\gamma^2cy^2+\gamma^3dy^3+\ldots\ldots+\gamma^{n-1}ky^{n-1}\\
&\&c.
\end{aligned}$$

from which the n unknown magnitudes a, b, c, d, ... k, must now be determined.

2. If we multiply these equations, in the order in which they stand vertically, first by the powers 1, α^n, β^n, γ^n, &c., then by the powers 1, α^{n-1}, β^{n-1}, γ^{n-1}, &c., after this by the powers 1, α^{n-2}, β^{n-2}, γ^{n-2}, &c., and so on; lastly by 1, α, β, γ, &c., and add the n results obtained by every such multiplication together; we then

get, since $'[n] = n$, and every numerical expression $'[p]$, whose radical exponent p is not divisible by $n=0$,

$$\begin{aligned}
na &= x' + x'' + x''' + x^{\prime v} + \&c.\\
nyb &= x' + \alpha^{n-1}x'' + \beta^{n-1}x''' + \gamma^{n-1}x^{\prime v} + \&c.\\
ny^2c &= x' + \alpha^{n-2}x'' + \beta^{n-2}x''' + \gamma^{n-2}x^{\prime v} + \&c.\\
ny^3d &= x' + \alpha^{n-3}x'' + \beta^{n-3}x''' + \gamma^{n-3}x^{\prime v} + \&c.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
ny^{n-2}i &= x' + \alpha^2x'' + \beta^2x''' + \gamma^2x^{\prime v} + \&c.\\
ny^{n-1}k &= x' + \alpha x'' + \beta x''' + \gamma x^{\prime v} + \&c.
\end{aligned}$$

3. Hence now we may immediately determine a; for, since $x' + x'' + x''' + x^{\prime v} + \&c. = [1] = A$, we then obtain from the first equation

$$a = \frac{A}{n}$$

Consequently a has only one single value; on the other hand, all the remaining magnitudes $b, c, d, \ldots\ldots i, k$, have more values than one, which may be obtained from the $n-1$ following equations, by transposing their roots $x', x'', x''', \ldots\ldots x^{(n)}$ in all possible ways. The equation on which each of them depends, is ∴ generally assumed to be of the $1 . 2 . 3 \ldots\ldots n$th degree. But if amongst these values there should be any equal ones, or any relation amongst them, then the degree of the equations can be reduced.

4. Since one of the $n+1$ magnitudes $a, b, c, d, \ldots k, p$, may be assumed to be arbitrary, we shall put $p = 1$; ∴ also $y = 1$. At the same time, in order to make the formulæ more simple, we shall put

$$k=\frac{a'}{n},\ i=\frac{a''}{n},\ h=\frac{a'''}{n},\ \ldots\ldots\ldots\ b=\frac{a^{(n-1)}}{n}$$

By these means, if we omit the first, and set down the others in an inverted order, the equations in 2 are transformed into the following ones:

$$\begin{aligned}
a' &= x' + \alpha x'' + \beta x''' + \gamma x^{\prime v} + \&c.\\
a'' &= x' + \alpha^2 x'' + \beta^2 x''' + \gamma^2 x^{\prime v} + \&c.\\
a''' &= x' + \alpha^3 x'' + \beta^3 x''' + \gamma^3 x^{\prime v} + \&c.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
a^{(n-1)} &= x' + \alpha^{n-1} x'' + \beta^{n-1} x''' + \gamma^{n-1} x^{\prime v} + \&c.
\end{aligned}$$

5. Since the roots of the equation $x^n - 1 = 0$ (because by the hypothesis n is a prime number) may be expressed by $1, \alpha, \alpha^2, \alpha^3, \ldots\ldots \alpha^{n-1}$, let α denote any imaginary root whatever; then we can put $\alpha^2, \alpha^3, \alpha^4$, &c. for β, γ, δ, &c.; by these means we obtain the first equation

$$a' = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \ldots\ldots + \alpha^{n-1} x^{(n)}$$

and the values of $a'', a''', a^{\prime v}, \ldots\ldots a^{(n-1)}$ are derived from this value of a', when we substitute $\alpha^2, \alpha^3, \alpha^4, \ldots\ldots \alpha^{n-1}$ for α, consequently when we substitute for α every imaginary root of the equation $y^n - 1 = 0$. If ∴ undetermined t denote each of the magnitudes $a', a'', a''', \ldots\ldots a^{(n-1)}$, then

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \ldots\ldots + \alpha^{n-1} x^{(n)}$$

6. Now, in order to find all the values of which t is capable, it is only necessary to permute the roots x', x'', x''', $\ldots\ldots x^{(n)}$, in all possible ways. For our purpose, however, it is more convenient, in this case, to proceed as follows: 1st, we only transpose the $n-2$ roots $x''', x^{\prime v}$,

x^{v}, ... $x^{(n)}$, while we let x', x'' retain their places, consequently we obtain $1 \cdot 2 \cdot 3$ ...... $n-2$ results; 2ndly, we put in each of the results thus obtained, first α^2, afterwards α^3, then α^4, and so on, and lastly α^{n-1} for α; then generally we have $1 \cdot 2 \cdot 3$ ...... $n-1$ results, which we should also have obtained, if we had let x' retain its place, and merely transposed the roots x'', x''', x^{IV} .... $x^{(n)}$; 3rdly and lastly, we multiply the $1 \cdot 2 \cdot 3$ ...... $n-1$ results so found, successively by $\alpha, \alpha^2, \alpha^3$, ... α^{n-1}, and we obtain, together with the former, the $1 \cdot 2 \cdot 3$ ...... n results, which arise from the transposition of all the roots x', x'', x''', ... $x^{(n)}$ and at the same time also all the values of t.

7. If $\therefore$ we denote the $1 \cdot 2 \cdot 3$ ...... $n-1$ values of t, which are obtained by the two first operations, by t', t'', t''', t^{v}, &c., then all the $1 \cdot 2 \cdot 3$ ...... n values of t may be expressed in the following way:

$$
\begin{array}{l}
t',\ \alpha t',\ \alpha^2 t',\ \alpha^3 t',\ \alpha^4 t',\ \ldots\ldots\ldots\ \alpha^{n-1} t' \\
t'',\ \alpha t'',\ \alpha^2 t'',\ \alpha^3 t'',\ \alpha^4 t'',\ \ldots\ldots\ldots\ \alpha^{n-2} t'' \\
t''',\ \alpha t''',\ \alpha^2 t''',\ \alpha^3 t''',\ \alpha^4 t''',\ \ldots\ldots\ldots\ \alpha^{n-1} t''' \\
\text{\&c.}
\end{array}
$$

If we put $t'^n = \theta'$, $t''^n = \theta''$, $t'''^n = \theta'''$ &c., then the values in the first horizontal series are the roots of the equation $t^n - \theta' = o$, those in the second series are the roots of the equation $t^n - \theta'' = o$, &c.; consequently the equation for t is the product of the equations

$$t^n - \theta' = o,\ t^n - \theta'' = o,\ t^n - \theta''' = o$$

&c.

8. Hence it follows, that the equation for t only contains such powers as are divisible by n. If $\therefore$ we put $t^n = \theta$, so that

$$\theta = (x' + ax'' + a^2x''' + a^3x^{IV} + \ldots\ldots + a^{n-1}x^{(n)})^n$$

we then obtain an equation for θ of the $1 . 2 . 3 \ldots n-1$th degree, whose roots are θ', θ'', θ''', &c. which we obtain from θ, by transposing the $n-1$, roots x'', x''' x^{IV},.....$x^{(n)}$ and allowing x to retain its place.

9. Instead of transposing the roots x'', x''', x^{IV}......$x^{(n)}$ in the expression θ, it will be quite sufficient, as in t, merely to permute the $n-2$ roots x''', x^{IV}, x^{V}, ... $x^{(n)}$, then substitute α^2, α^3, α^4, ... α^{n-1} for α, and in the $n-1$ values of θ thus obtained, transpose the roots x''', x^{IV}, x^{V}, ... $x^{(n)}$.

10. We now assume, that the $n-1$ values of θ, which arise from the substitution of α^2, α^3, α^4, ... α^{n-1} for α, are roots of the following equation:

$$(A) \ldots\ldots \theta^{n-1} - p\theta^{n-2} + q\theta^{n-3} - r\theta^{n-4} + \&c. = 0;$$

then the coefficients p, q, r, &c. are functions of these values, and can $\therefore$, as is the case also with these last, suffer no change by the transposition of x'. But since they are also symmetrical with respect to these values, consequently they can likewise undergo no change by the substitution of α^2, α^3, α^4, ... α^{n-1} for α, because the only consequence arising from this substitution is, that of the $n-1$ values of θ, one is merely transformed into the other. Hence, however, it follows, that these coefficients can have no more unequal values, than those

which arise exclusively from the transposition of the $n-2$ roots $x''', x^{IV}, x^{V}, \ldots\ldots x^{(n)}$, and that consequently they all depend on equations of the $1 \cdot 2 \cdot 3 \ldots\ldots n-2$th degree.

11. Therefore the equation for θ, which, as we have seen, is of the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$th degree, may always, when n is a prime number, be analyzed into $1 \cdot 2 \cdot 3 \ldots \ldots n-2$ equations of the $n-1$th degree. Now, if (A) be one of these equations, and $\theta', \theta'', \theta''' \ldots\ldots \theta^{(n-1)}$ its $n-1$ roots, then $\sqrt[n]{\theta'}, \sqrt[n]{\theta''}, \sqrt[n]{\theta'''}, \ldots\ldots \sqrt[n]{\theta^{(n-1)}}$, are the corresponding values of t, or of $a', a'', a''', \ldots\ldots \ldots a^{(n-1)}$, and if we substitute these values in any order in the expressions in 4, we then obtain

$$b = \frac{\sqrt[n]{\theta'}}{n}, c = \frac{\sqrt[n]{\theta''}}{n}, d = \frac{\sqrt[n]{\theta'''}}{n}, \text{ \&c.}$$

consequently, since we have put $p = 1$,

$$x = \frac{A}{n} + \frac{1}{n} - (\sqrt[n]{\theta'} + \sqrt[n]{\theta''} + \sqrt[n]{\theta'''} + \ldots + \sqrt[n]{\theta^{(n-1)}})$$

But that for θ only those of the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ values may be assumed, which belong to one and the same equation, it matters not which, $\theta^{n-1} - p\,\theta^{n-2} + \text{\&c.} = 0$, appears from this circumstance, that the $n-1$ values of t, consequently the corresponding values of θ must so depend upon one another, that we obtain them all, when in one of them we substitute $\alpha^2, \alpha^3, \alpha^4, \ldots\ldots \alpha^{n-1}$ for α, 5.

12. If we wish actually to find the equation (A), first of all we solve the function

$$\theta = (x' + \alpha x'' + \alpha^2 x''' + \ldots\ldots + \alpha^{n-1} x^{(n)})^n$$

according to the powers of α, which may very easily be done by means of the polynomial theorem. But since α^n, α^{n+1}, α^{n+2}, &c. are no other than 1, α, α^2, &c. consequently this development, after the proper reduction, takes the following form:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \ldots\ldots + \xi^{(n)}\alpha^{n-1}$$

and ξ', ξ'', ξ''', ... $\xi^{(n-1)}$ are mere functions of x', x'', x''', ... $x^{(n-1)}$ without α. If in this value of θ we put α^2, α^3, α^4, ... α^{n-1} successively for α, or, which is the same, β, γ, δ, &c. for α, we then obtain the values of θ', θ'', θ''', ... $\theta^{(n-1)}$, viz.:

$$\begin{aligned}
\theta' &= \xi' + \xi''\alpha + \xi'''\alpha^2 + \ldots\ldots + \xi^{(n)}\alpha^{n-1} \\
\theta'' &= \xi' + \xi''\beta + \xi'''\beta^2 + \ldots\ldots + \xi^{(n)}\beta^{n-1} \\
\theta''' &= \xi' + \xi''\gamma + \xi'''\gamma^2 + \ldots\ldots + \xi^{(n)}\gamma^{n-1}
\end{aligned}$$

&c.

from which the equation (A) may be compounded in the usual way. The coefficients p, q, r, &c. are then still functions of the roots x', x'', x''', ... $x^{(n)}$, but such, that they only change when the roots x''', x^{IV}, x^{V}, ... $x^{(n)}$ are transposed. The equations for these functions may be found by the method given in the third chapter. Moreover it is quite sufficient, as will be shown in the following chapter, to find one of these equations, for instance, that for p, because from the known value of p, the values of q, r, &c. may be found directly, and without the solution of any other equation. It is likewise sufficient, as follows from 11, to find only a single value for each of the coefficients p, q, r, &c.

13. From all that has hitherto been said, it follows, that the solution of an equation of the nth degree, when n is a prime number, depends on the solution of an equation for p of the $1 . 2 . 3 \ldots n-2$th degree; and this result coincides with that which has been found by another method in § CXXXVI. With the view to further elucidation, I shall now apply it to an equation of the fifth degree.

REMARK. When n is a compound number, then the conclusions drawn with respect to the substitution of the root α for α^2, α^3, $\alpha^4, \ldots\ldots \alpha^{n-1}$, suffer the same changes, as the conclusions, § CXXXVI, § CXXXVII, must undergo in the same case, and we shall then find, as we did there, that an equation of the nth degree leads to an equation of the $\frac{1 . 2 . 3 \ldots n-1}{\lambda}$th degree, when λ denotes the number of the primitive roots of the equation $y^n - 1 = 0$.

SECTION CXL.

PROB. From the given equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

find actually the reduced equation for the magnitude p of the foregoing §.

Solution 1. The equation (A) in 10 of the foregoing §, when $n = 5$, is here

$$\theta^4 - p\theta^3 + q\theta^2 - r\theta + s = 0$$

and the roots of this equation are, when α, β, γ, δ, denote the imaginary roots of the equation $y^5-1=0$,

$$\begin{aligned}
\theta' &= \xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \xi^{V}\alpha^4\\
\theta'' &= \xi' + \xi''\beta + \xi'''\beta^2 + \xi^{IV}\beta^3 + \xi^{V}\beta^4\\
\theta''' &= \xi' + \xi''\gamma + \xi'''\gamma^2 + \xi^{IV}\gamma^3 + \xi^{V}\gamma^4\\
\theta^{IV} &= \xi' + \xi''\delta + \xi'''\delta^2 + \xi^{IV}\delta^3 + \xi^{V}\delta^4
\end{aligned}$$

2. If we add these equations together, we then obtain, when $\theta'+\theta''+\theta'''+\theta^{IV}=p$, and $\alpha+\beta+\gamma+\delta={}^{\backprime}[1]-1=-1$, $\alpha^2+\beta^2+\gamma^2+\delta^2={}^{\backprime}[2]-1=-1$, $\alpha^3+\beta^3+\gamma^3+\delta^3={}^{\backprime}[3]-1=-1$, $\alpha^4+\beta^4+\gamma^4+\delta^4={}^{\backprime}[4]-1=-1$;

$$\begin{aligned}
p &= 4\xi' - (\xi'' + \xi''' + \xi^{IV} + \xi^{V})\\
&= 5\xi' - (\xi' + \xi'' + \xi''' + \xi^{IV} + \xi^{V})
\end{aligned}$$

3. The second part of the expression for p, viz. $\xi'+\xi''+\xi'''+\xi^{IV}+\xi^{V}$, may be found immediately. For from 12 of the foregoing § it follows, that the development of

$$(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V})^5$$

assumes the following forms:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \xi^{V}\alpha^4;$$

and this form of the development is always correct, whichever root of the equation $x^5-1=0$ we substitute for α; consequently also, when we put $\alpha=1$. If this is actually done, we find

$$\begin{aligned}
&\xi' + \xi'' + \xi''' + \xi^{IV} + \xi^{V} =\\
&(x' + x'' + x''' + x^{IV} + x^{V})^5\\
&= [1]^5 = A^5
\end{aligned}$$

We have ∴ also

$$p = 5\xi' - A^5.$$

4. In order ∴ to determine p, it is only necessary to find ξ', consequently that term in the development of $(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V})^5$, which does not contain α. For this purpose we can give this expression the following form:

$$\alpha^{-5}\,(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + \alpha^5 x^{V})^5$$

or, since $\alpha^{-5} = 1$, the following one

$$(\alpha x + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + \alpha^3 x^{V})^5$$

from which we derive this advantage, that the dashes over x coincide with the exponents of α. For now we have nothing more to do, as is shown in the polynomial theorem, but to combine the roots x', x'', x''', x^{IV}, x^{V}, in all possible ways, in such a way that the sum of the dashes $=5$, $=10$, $=15$, $=20$, $=25$, because $\alpha^5 = \alpha^{10} = \alpha^{15} = \alpha^{20} = \alpha^{25} = 1$. In this way we find, when $[5]$, $[1^5]$, are substituted for ${x'}^5 + {x''}^5 + {x'''}^5 + {x^{IV}}^5 + {x^{V}}^5$, $x'x''x'''x^{IV}x^{V}$,

$$\xi' = [5] + [1^5] +$$

$$20\left\{\begin{array}{l} {x'}^3 x'' x^{V} + {x'}^3 x''' x^{IV} + x' {x''}^3 x''' + x' x'' {x^{IV}}^3 + \\ x' {x'''}^3 x^{V} + {x''}^3 x^{IV} x^{V} + x'' {x'''}^3 x^{IV} + x' x^{IV} {x^{V}}^3 \\ \qquad + x'' x''' {x^{V}}^3 + x''' {x^{IV}}^3 x^{V} \end{array}\right\}$$

$$+30\left\{\begin{array}{l} {x'}^2 {x''}^2 x^{IV} + {x'}^2 x'' {x'''}^2 + {x'}^2 x''' {x^{V}}^2 + {x'}^2 {x^{IV}}^2 x^{V} + \\ x' {x''}^2 {x^{V}}^2 + x' {x'''}^2 {x^{IV}}^2 + {x''}^2 {x'''}^2 x^{V} + {x''}^2 x''' {x^{IV}}^2 \\ \qquad + x'' {x^{IV}}^2 {x^{V}}^2 + {x'''}^2 x^{IV} {x^{V}}^2 \end{array}\right\}$$

or when, for shortness' sake, in the value of ξ', we denote by ζ that which is not to be found in the crotchets we get

$$\xi' = [5] + [1^5] + \zeta$$

∴

$$p = 5\zeta + 5[5] + 5[1^5] - A^5$$

5. Amongst the 120 values, which the function ζ contains by the transposition of the roots x', x'', x''', x^{IV}, x^{V},

we find no more than six unequal ones, and they will be exactly those which arise exclusively from the transposition of the three roots x''', x'^{v}, x^{v}. If we denote these values by ζ', ζ'', ζ''', ξ'^{v}, ζ^{v}, $\zeta^{v'}$, and the corresponding values of p by p', p'', p''', p'^{v}, p^{v}, $p^{v'}$, we then obtain

$$\begin{aligned}
p' &= 5\,\zeta' + 5\,[5] + 5\,[1^5] - A^5 \\
p'' &= 5\,\zeta'' + 5\,[5] + 5\,[1^5] - A^5 \\
p''' &= 5\,\zeta''' + 5\,[5] + 5\,[1^5] - A^5 \\
p'^{v} &= 5\,\zeta'^{v} + 5\,[5] + 5\,[1^5] - A^5 \\
p^{v} &= 5\,\zeta^{v} + 5\,[5] + 5\,[1^5] - A^5 \\
p^{v'} &= 5\,\zeta^{v'} + 5\,[5] + 5\,[1^5] - A^5
\end{aligned}$$

and these six values of p are the roots of the required equation. We already know, from the third chapter, how to proceed further, in order to find this equation itself. It would be better, however, instead of the equation for p to find that for ζ; for if we have ζ, we have also p.

VII.—A GENERAL METHOD BY WHICH, FROM THE KNOWN VALUE OF A GIVEN FUNCTION OF THE ROOTS OF AN EQUATION, TO FIND THE VALUE OF EVERY OTHER FUNCTION OF THESE ROOTS.

SECTION CXLI.

ALL the methods which we have hitherto applied to the solution of equations, are founded either on analysis or transformation. The first, from its very nature, cannot be general, because every equation will not admit of being analyzed into others of lower degrees. Consequently, in the general solution of equations, we have no other mode left us but to transform the given equations into others, which in themselves are either solvible by the methods already known, or may be made so by analysis.

Now, let it be assumed that we have transformed in any way, no matter which, the given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

into another

$$t^m + A't^{m-1} + B't^{m-2} + C't^{m-3} + \&c. = 0$$

then the roots of the last equation must stand in some one relation to the roots of the first, or, in other words, t must admit of being expressed by some function of the roots x', x'', x''', &c. Now I affirm, that it is always

allowable to assume t to be a rational function of these roots. For, let $F: (x') (x'') (x''') \dots (x^{(n)})$ be any irrational function of these roots, and let $t = F: (x') (x'') (x''') \dots (x^{(n)})$; then this equation, as has been already shown in the fifth chapter, can always be made rational by removing the irrational magnitudes. We shall thus get an equation

$$t^{\mu} + A''t^{\mu-1} + B''t^{\mu-2} + C''t^{\mu-3} + \&c. = 0$$

in which the coefficients A'', B'', C'', &c. are all rational functions of x', x'', x''', &c. Now if we eliminate from this equation and the equation $t^{m} + A't^{m-1} + B't^{m-2} + \&c. = 0$ all the powers of t, as far as the first, we then obtain for t a rational function only.

In the first place ∴ it is only necessary, in the transformation of the equations, to find such rational functions of x', x'', x''', &c., for which the transformed equation is either immediately solvible, or at least may be made to depend on solvible equations. But this is not all; it is not sufficient to know the values of the assumed function; we must also be able, from these values, to find the roots x', x'', x''', &c. I shall first handle the second subject, and, according to Mr. Lagrange, in the third volume of the New Berlin Memoirs, show the method by which, from the known values of a given function, the value of every other function may be found, consequently also the roots themselves. Here two cases must be taken into consideration, viz. first, the case in which the given and the required function are homogeneous; secondly, the case in which they are not so.

For the sake of greater perspicuity, when I treat of the

values of a function, I shall sometimes distinguish the values of forms from numerical values; the first are the different forms themselves, which arise from the transposition of the roots x', x'', x''', &c.; the latter, the actual values of these forms expressed by given magnitudes.

SECTION CXLII.

Prob. Let it be assumed that the given equation

$$\text{I. } x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

by the introduction of a new magnitude $t = f: (x') (x'') (x''')...(x^{(n)})$, according to the method in the third chapter, is transformed into an equation

$$\text{II. } t^\pi + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + ... + U = 0$$

which is completely solvible, consequently all of whose roots may be found: from these known numerical values of the function t, it is required to find the numerical values of any other function $y = \phi: (x') (x'') (x''') ...(x^{(n)})$ respecting which it is assumed that it is homogeneous to the former.

Solution 1. Since the functions t, y, according to the hypothesis, are homogeneous, then, by the transposition of the roots x', x'', x''', &c. the former must contain exactly as many unequal values as the latter. The function t, however, has π values, because the equation II, by which it is represented, has been assumed to be of the πth degree, consequently the second function has also π values. I shall denote the values of forms of t by t', t'', t''', $...t^{(\pi)}$, and the values of forms of y by y', y'', y''', $...y^{(\pi)}$,

and assume besides, that those which have the same number of dashes, arise from the same transpositions.

2. Since t and y are homogeneous functions, consequently any expression whatever, which is compounded of these functions, can have no more unequal values of forms than these functions themselves. Consequently, also, such an expression as $t^{\lambda}y$ can have no more than π different values, and these are $t'^{\lambda}y'$, $t''^{\lambda}y''$, $t'''^{\lambda}y''' + \ldots\ldots +$ $(t^{(\pi)\lambda}y^{(\pi)})$. If we take the sum of all these values, we obtain the function

$$(\psi)\ldots\ldots t'^{\lambda}y' + t''^{\lambda}y'' + t'''^{\lambda}y''' + \ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$$

and this function has the property of remaining the same, however we transpose the roots x', x'', x''', &c.; it is ∴ symmetrical with respect to the former roots, and consequently, let λ be any number whatever, may always be expressed rationally by the coefficients A, B, C, &c. of the given equation.

3. If we denote the numerical values, which the function (ψ) contains, when we substitute $0, 1, 2, 3, \ldots \pi-1$ sucessively for λ, by $z_0, z_1, z_2, z_3, \ldots\ldots z_{\pi-1}$, we obtain the following π equations:

$$\begin{aligned}
y' + y'' + y''' + \ldots\ldots + y^{(\pi)} &= z_0 \\
t'y' + t''y'' + t'''y''' + \ldots\ldots + t^{(\pi)}y^{(\pi)} &= z_1 \\
t'^2y' + t''^2y'' + t'''^2y''' + \ldots\ldots + (t^{(\pi)})^2y^{(\pi)} &= z_2 \\
t'^3y' + t''^3y'' + t'''^3y''' + \ldots\ldots + (t^{(\pi)})^3y^{(\pi)} &= z_3 \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
t'^{\pi-1}y' + t''^{\pi-1}y'' + t'''^{\pi-1}y''' + \ldots + (t^{(\pi)})^{\pi-1}y^{(\pi)} &= z_{\pi-1}
\end{aligned}$$

in which $z_0, z_1, z_2, z_3, \ldots\ldots z_{\pi-1}$, are all known magnitudes expressed by the coefficients of the equation I.

4. Now, let t', t'', t''', ... $t^{(\pi)}$, instead of the values of forms of the function, denote its numerical values, then these are no other than the roots of the equation II, consequently, by the hypothesis, are all known. Therefore in the foregoing π equations there are no other unknown magnitudes but y', y'', y''',...$y^{(\pi)}$; and since their number is π, consequently we have exactly the same number as of equations; they may ∴, with a few exceptions (which will be inquired into hereafter), always be calculated and expressed rationally by the magnitudes t', t'', t''' ...... $t^{(\pi)}$ and z_0, z_1, z_2, z_3, ... $z_{\pi-1}$, consequently also by the magnitudes t', t'', t''' ...... $t^{(\pi)}$, and the coefficients A, B, C, &c. of the given equation.

Example. When $\pi = 1$, we only have

$$y' = z_0$$

which must also be the case, because then t and y are symmetrical functions of x', x'', x''', &c. and ∴ y no longer depends on t, but only on the coefficients A, B, C, &c.

When $\pi = 2$, we have the two equations

$$y' + y'' = z_0$$
$$t'y' + t''y'' = z_1$$

and hence

$$y' = \frac{z_1 - t''z_0}{t' - t''},\ y'' = \frac{z_1 - t'z_0}{t'' - t'}$$

When $\pi = 3$, we have the three equations

$$y' + y'' + y''' = z_0$$
$$t'y' + t''y'' + t'''y''' = z_1$$
$$t'^2y' + t''^2y'' + t'''^2y''' = z_2$$

and hence we obtain

$$y' = \frac{z_2 - (t'' + t''')z_1 + t''t'''z_0}{(t' - t'')\,(t' - t''')}$$

$$y'' = \frac{z_2 - (t' + t''')z_1 + t't'''z_0}{(t'' - t')\,(t'' - t''')}$$

$$y''' = \frac{z_2 - (t' + t'')z_1 + t't''z_0}{(t''' - t')\,(t''' - t'')}$$

In the same way, when $\pi = 4$, we find the following values for $y', y'', y''', y^{\prime v}$.

$$\frac{z_3 - (t'' + t''' + t^{\prime v})z_2 + (t''t''' + t''t^{\prime v} + t'''t^{\prime v})z_1 - t''t'''t^{\prime v}z_0}{(t' - t'')\,(t' - t''')\,(t' - t^{\prime v})}$$

$$\frac{z_3 - (t' + t''' + t^{\prime v})z_2 + (t't''' + t't^{\prime v} + t'''t^{\prime v})z_1 - t't'''t^{\prime v}z_0}{(t'' - t')\,(t'' - t''')\,(t'' - t^{\prime v})}$$

$$\frac{z_3 - (t' + t'' + t^{\prime v})z_2 + (t't'' + t't^{\prime v} + t''t^{\prime v})z_1 - t't''t^{\prime v}z_0}{(t''' - t')\,(t''' - t'')\,(t''' - t^{\prime v})}$$

$$\frac{z_3 - (t' + t'' + t''')z_2 + (t't'' + t't''' + t''t''')z_1 - t't''t'''z_0}{(t^{\prime v} - t')\,(t^{\prime v} - t'')\,(t^{\prime v} - t''')}$$

from which the law of the progression may be very easily seen.

SECTION CXLIII.

Prob. All that has been said in the problem in the foregoing § holds, with this single difference, that all the roots of the equation II, as was there assumed, are not known, but merely one of them: required now to find the numerical value of the function y corresponding to this numerical value of the function t.

Solution 1. Let t' be the known root of the equation II. If we divide this equation by $t - t'$, we obtain another equation

III. $t^{\pi-1} + p' t^{\pi-2} + Q' t^{\pi-3} + \ldots\ldots\ldots + U' = o$.
in which

$$P' = t' + P$$
$$Q' = t'^2 + Pt' + Q$$
$$R' = t'^3 + Pt'^2 + Qt' + R$$
$$S' = t'^4 + Pt'^3 + Qt'^2 + Rt' + S$$
&c.

and the roots of this equation are t'', t''', t'^{V}, ... $t^{(\pi)}$.

2. But since in this case the single root t' was assumed to be known, we must merely endeavour to express y' by t'; and this object is most easily attained in the following way by means of the method of elimination given in § LVIII. Multiply the equations in 3 of the foregoing §, beginning with the last but one, and proceeding upwards, by P', Q', R', &c. viz. the last but one by P', the one preceding it by Q', and so on to the first, which is multiplied by U', and then add the results thus obtained to the last equation; by these means we obtain

$$\begin{aligned} & z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0 \\ = & y' \, (t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \ldots\ldots + U') \\ + & y'' \, (t''^{\pi-1} + P't''^{\pi-2} + Q't''^{\pi-3} + \ldots\ldots + U') \\ + & y''' \, (t'''^{\pi-1} + P't'''^{\pi-2} + Q't'''^{\pi-3} + \ldots\ldots + U') \end{aligned}$$
&c.

3. Since t'', t''', t'^{V}, ... $t^{(\pi)}$, are the roots of the equation III, then all that which has been multiplied by y'', y''', y'^{V}, ... $y^{(\pi)}$ in the second part of the equation just found, $=$ o. We only ∴ retain

$$\begin{aligned} & z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0 \\ = & y' \, (t^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \ldots\ldots + U') \end{aligned}$$

and hence it follows

$$y' = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \dots\dots + U'z_0}{t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \dots\dots + U'}$$

4. If in this we substitute for t' every other root of the equation II, we then obtain the numerical values of y'', y''', y'_v, ... $y^{(\pi)}$. If $\therefore$ t and y which are undetermined, denote two corresponding values of the functions just given, we then have generally

$$y = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \dots\dots + U'z_0}{t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \dots\dots + U'}$$

and it is then

$$\begin{aligned} P' &= t + P \\ Q' &= t^2 + Pt + Q \\ R' &= t^3 + Pt^2 + Qt + R \\ &\&c. \end{aligned}$$

Example. In § XXXIX we find, that when $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ is the given equation, the function $t = x'x'' + x'''x^{iv}$ depends on the following equation of the third degree:

$$t^3 - Bt^2 + (AC - 4D)\,t - (C^2 - 4BD + A^2D) = 0.$$

I shall now assume, that we have so far solved this equation, that we have found one of its roots, and that we now wished to determine from it the value of another function $y = (x'x'' - x'''x^{iv})^2$, which is homogeneous to the former.

Since here $\pi = 3$, we then have

$$y = \frac{z_2 + P'z_1 + Q'z_0}{t^2 + P't + Q'}$$

Further, since $P = -B$, $Q = AC - 4D$, we have

$$P' = t + P = t - B$$
$$Q' = t^2 + Pt + Q = t^2 - Bt + AC - 4D$$

It only remains now to determine the values of z_0, z_1, z_2. But

$$t' = x'x'' + x'''x^{\text{v}},\quad y' = (x'x'' - x'''x^{\text{v}})^2$$
$$t'' = x'x''' + x''x^{\text{v}},\quad y'' = (x'x''' - x''x^{\text{v}})^2$$
$$t''' = x'x^{\text{v}} + x''x''',\quad y''' = (x'x^{\text{v}} - x''x''')^2$$

which, when we take the numerical expressions from the annexed Tables, give the following values:

$$z_0 = y' + y'' + y''' = [2^2] - 6[1^4]$$
$$= B^2 - 2AC - 4D$$
$$z_1 = t'y' + t''y'' + t'''y''' = [3^2] - [1^2 2^2]$$
$$= B^3 - 3ABC + 3C^2 + 3A^2D - 4BD$$
$$z_2 = t'^2y' + t''^2y'' + t'''^2y''' = [4^2] - 6[2^4]$$
$$= B^4 - 4AB^2C + 2A^2C^2 + 4BC^2 + 4A^2BD$$
$$- 4B^2D - 8ACD$$

If we substitute the values of P', Q', z_0, z_1, z_2 here found, we obtain

$$y = \frac{\begin{pmatrix}(B^2-2AC-4D)t^2-(ABC-3C^2-3A^2D)t\\ +16D^2-4B^2D+BC^2+A^2BD-4ACD\end{pmatrix}}{3t^2 - 2Bt + AC - 4D}$$

and by means of this expression we are now enabled, for each numerical value of the function t, to find a numerical value of the function y.

Remark. By means of the differential calculus, we can give the denominator of the general expression for y in 4 a more simple form. Thus, since

$$(t-t')\,(t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \ldots\ldots + U')$$
$$= (t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + \ldots\ldots + U)$$

we then have, when we differentiate both sides, in reference to t, and divide by the differential dt,

$$(t-t') \left[(\pi-1)t^{\pi-2}+(\pi-2)\, P't^{\pi-3}+(\pi-3)\, Q't^{\pi-4}+\&c.\right] + t^{\pi-1}+P't^{\pi-2}+Q't^{\pi-3}+R't^{\pi-4}+\&c.$$

$$=$$

$$\pi t^{\pi-1}+(\pi-1)Pt^{\pi-2}+(\pi-2)Qt^{\pi-3}+(\pi-3)\, Rt^{\pi-4}+\&c.$$

If in these we substitute t' for t, we obtain the equation

$$t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + R't'^{\pi-4} + \&c.$$
$$=\pi t'^{\pi-1} + (\pi-1)\, Pt'^{\pi-2} + (\pi-2)\, Qt'^{\pi-3} + \&c.$$

and since this equation must be correct, whatever root we assume for t', we then have generally

$$t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \&c. =$$
$$\pi t^{\pi-1}+(\pi-1)\, Pt^{\pi-2}+(\pi-2)\, Qt^{\pi-3}+(\pi-3)Rt^{\pi-4}+\&c.$$

Consequently the value of y may also be expressed in the following way

$$y = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \dots\dots\dots\dots\dots\dots + U'z_0}{\pi t^{\pi-1} + (\pi-1)\, Pt^{\pi-2} + (\pi-2)\, Qt^{\pi-3} + \dots + T}$$

SECTION CXLIV.

If the formula of the foregoing § be generally applicable, we are enabled, from the given value of any function $f: (x') \, (x'') \, (x''') \dots\dots (x^{(n)})$ to find the value of every other function $\phi: \ (x') \, (x'') \, (x''') \dots\dots (x^{(n)})$, homogeneous to it, and that immediately merely by a rational expression. But it is also actually applicable in all imaginable cases, with the single exception of the one in which the value of t is such, that the denominator of the expression for $y = o$; a case which was mentioned in § LX. In order to see how the case is here, I shall consider the denominator $t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3}+\&c.$

in the expression for y in 3 of the foregoing §. It is, from its origin, no other than the product of the factors $t' - t''$, $t' - t'''$, $t' - t^{\text{v}}$, ... $t' - t^{(\pi)}$. If ∴ it vanishes, then one or other of these factors $= 0$, and t' ∴ must be equal to one, or even more of the roots t'', t''', t^{v}, ... $t^{(\pi)}$. Hence it follows, that the case in which the denominator in the expression for y vanishes, can only obtain, when the equation II has equal roots. But now it may likewise be seen, why this expression cannot give the value of y'. For so long as a number of roots t', t'', t''', ... $t^{(\nu)}$ are different from one another, t' gives the value of y', t'' the value of y'', &c. But if they are equal to one another, then the single root t' must at once give the ν values y', y'', y''', ... $y^{(\nu)}$; but since the expression found for y is rational, this is impossible. Hence it may be further concluded, that the ν values y', y'', y''', ... $y^{(\nu)}$ must be given by a single irrational expression, which contains exactly ν values, or, which is the same, that they must depend on an equation of the nth degree, whose coefficients are all rational. How this equation may be found, will be seen immediately.

SECTION CXLV.

Auxiliary Rule.

PROB. Let Π denote any function of x, and let the equation

$$y = (x - a)^m \, \Pi$$

be given: required to find the value of the differential proportional $\frac{d^m y}{dx^m}$ for the case, where $x = a$.

Solution 1. Let $m = 1$; $\therefore y = (x-a)\,\Pi$. If we differentiate this equation, we find

$$dy = (x-a)\,d\Pi + \Pi dx$$

If in this equation we put $x = a$, then the first term of the second part vanishes, and we consequently have, when Π' denotes what Π becomes when we put $x = a$, $dy = \Pi' dx$, and $\frac{dy}{dx} = \Pi'$.

2. Let $m = 2$; $\therefore y = (x-a)^2\Pi$. If we differentiate this equation twice successively, we find

$$dy = (x-a)^2\,d\Pi + 2\,(x-a)\,\Pi dx$$
$$d^2y = (x-a)^2\,d^2\Pi + 4\,(x-a)\,d\Pi dx + 1\,.\,2\;\Pi dx^2$$

If we put $x = a$ in the second equation, the two first terms of the second part vanish, and we then have $d^2y = 1\,.\,2\,\Pi' dx^2$; consequently $\frac{d^2y}{dx^2} = 1\,.\,2\,\Pi'$.

3. Let $m = 3$; $\therefore y = (x-a)^3\,\Pi$. If we differentiate this equation three times in succession, we then obtain successively,

$$dy = (x-a)^3 d\,\Pi + 3\,(x-a)^2\Pi dx$$
$$d^2y = (x-a)^3 d^2\Pi + 6\,(x-a)^2 d\Pi dx + 2\,.\,3\,(x-a)\Pi dx^2$$
$$d^3y = (x-a)^3 d^3\Pi + 9\,(x-a)^2 d^2\Pi dx + 18\,(x-a) d\Pi dx^2 + 1\,.\,2\,.\,3\;\Pi dx^3$$

and when we put $x = a$, $d^3y = 1\,.\,2\,.\,3\;\Pi' dx^3$, $\therefore$ $\frac{d^3y}{dx^3} = 1\,.\,2\,.\,3\Pi'$.

4. Generally, as is easily seen from the continuation of the calculation, we find for $d^m y$, after differentiating the

equation $y=(x-a)^m\Pi$ m times, a differential expression, whose last term is $1 \,.\, 2 \,.\, 3 \ldots m\Pi dx^m$, and in which all the remaining terms contain the factor $x-a$. If $\therefore$ we put $x=a$, we then obtain $d^m y = 1 \,.\, 2 \,.\, 3 \ldots\ldots m\Pi' dx^m$, and consequently

$$\frac{d^m y}{dx^m} = 1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots m\Pi'$$

SECTION CXLVI.

PROB. When t and y denote two homogeneous functions of the roots x', x'', x''', &c. of the given equation

$$\text{I. } x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = o$$

from the known value of the function t it is required to find the value of the function y in the case where the equation

$$\text{II. } t^\pi + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + \&c. = o$$

on which the first depends, contains equal roots, amongst which is the known value of t.

Solution 1. In the remark in § CXLIII, we find the following expression for y:

$$y = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots\ldots + U'z_0}{\pi t^{\pi-1} + (\pi-1)\,Pt^{\pi-2} + (\pi-2)\,Qt^{\pi-3} + \&c.}$$

in which $P' = t + P$, $Q' = t^2 + Pt + Q$, &c.; and from this general expression we obtain the particular values of y', y'', y''', &c., when we substitute t', t'', t''', &c. for t. Now we wish, in the first place, to give this expression a form which will be more convenient for our purpose.

2. Since t', t'', t''', &c. are the roots of the equation II, then

$$t^\pi + Pt^{\pi-1} + Qt^{\pi-2} + \&c. =$$
$$(t-t')\,(t-t'')\,(t-t''')\,(t-t^{\prime v})\ldots\ldots(t-t^{(\pi)})$$

If we differentiate this equation in reference to t, we obtain, after dividing by dt,

$$\pi t^{\pi-1} + (\pi-1)\,Pt^{\pi-2} + (\pi-2)\,Qt^{\pi-3} + \&c. =$$
$$(t-t'')\;(t-t''')\;(t-t^{\prime v})\;\ldots\ldots\ldots\ldots\;(t-t^{(\pi)})$$
$$+\;(t-t')\;(t-t''')\;(t-t^{\prime v})\;\ldots\ldots\ldots\ldots\;(t-t^{(\pi)})$$
$$+\;(t-t')\;(t-t'')\;(t-t^{\prime v})\;\ldots\ldots\ldots\ldots\;(t-t)^{(\pi)})$$
$$\&c.$$

If in this we substitute t', t'', t''', &c. successively for t, we obtain

$$\pi t'^{\pi-1} + (\pi-1)\,Pt'^{\pi-2} + (\pi-2)\,Qt'^{\pi-3} + \&c. =$$
$$(t'-t'')\;(t'-t''')\;(t'-t^{\prime v})\;\ldots\ldots\ldots\;(t'-t^{(\pi)})$$
$$\pi t''^{\pi-1} + (\pi-1)\,Pt''^{\pi-2} + (\pi-2)\,Qt''^{\pi-3} + \&c. =$$
$$(t''-t')\;(t''-t''')\;(t''-t^{\prime v})\;\ldots\ldots\ldots\;(t''-t^{(\pi)})$$
$$\pi t'''^{\pi-1} + (\pi-1)\,Pt'''^{\pi-2} + (\pi-2)\,Qt'''^{\pi-3} + \&c. =$$
$$(t'''-t')\;(t'''-t'')\;(t'''-t^{\prime v})\;\ldots\ldots\;(t'''-t^{(\pi)})$$
$$\&c.$$

3. Now, if we denote that which the numerator of the expression for t becomes by the substitution of t', t'', t''', &c. for t, by Ω', Ω'', Ω''', &c., we then obtain, by means of the results in 2,

$$y' = \frac{\Omega'}{(t'-t'')\;(t'-t''')\;(t'-t^{\prime v})\ldots\ldots(t'-t^{(\pi)})}$$
$$y'' = \frac{\Omega''}{(t''-t')\;(t''-t''')\;(t''-t^{\prime v})\ldots\ldots(t''-t^{(\pi)})}$$
$$y''' = \frac{\Omega'''}{(t'''-t')\;(t'''-t'')\;(t'''-t^{\prime v})\ldots\ldots(t'''-t^{(\pi)})}$$
$$\&c.$$

From the form of these values it is evident, that when $t' = t''$, the denominators in the values of y' and y'', are both $= 0$; whence, by § CXLIV, it may be concluded, that these values cannot be determined singly by a rational expression, but depend on an equation of the second degree. In like manner, when we put $t' = t'' = t'''$, the denominators in the values of y', y'', y''', all vanish, and consequently in this case these values must depend on a single equation of the third degree; and in a similar way it holds, when more values of t are equal to one another.

4. In the first place we assume, that the equation II has no more than two equal roots t', t''. First let them be unequal, and let them differ by an infinitely small magnitude h, so that $t'' = t' + h$. Further, let, for shortness' sake,

$$(t' - t''')\,(t' - t^{\mathrm{v}})\,\ldots\ldots\ldots(t' - t^{(\pi)}) = \Pi'$$
$$(t'' - t''')\,(t'' - t^{\mathrm{v}})\,\ldots\ldots\ldots(t'' - t^{(\pi)}) = \Pi''$$

we then have

$$y' = \frac{\Omega'}{(t' - t'')\,\Pi'} = \frac{\Omega'}{-h\Pi'}$$
$$y'' = \frac{\Omega''}{(t'' - t')\,\Pi''} = \frac{\Omega''}{h\Pi''}$$

and ∴

$$y' + y'' = \frac{1}{h}\left[\frac{\Omega''}{\Pi''} - \frac{\Omega'}{\Pi'}\right]$$

5. If we omit the infinitely small magnitude in this last equation, we then can put $\Pi'' = \Pi'$, and we consequently have

$$y' + y'' = \frac{\Omega'' - \Omega'}{h} \cdot \frac{1}{\Pi'}$$

But according to Taylor's Theorem

$$\Omega'' - \Omega' = \frac{d\Omega'}{dt'} \cdot \frac{h}{1} + \frac{d^2\Omega'}{dt'^2} \cdot \frac{h^2}{1 \cdot 2} + \&c.$$

If $\therefore$ we divide this expression by h, and then put $h = 0$, we obtain

$$y' + y'' = \frac{d\Omega'}{dt'} \cdot \frac{1}{\Pi'}$$

6. We now assume, that the equation II has three equal roots, which are t', t'', t'''. As before, consider again these roots at first as differing, by an infinitely small magnitude, and put $t'' = t' + h$, $t''' = t' + k$; further, put

$$(t' - t^{IV})\,(t' - t^{V})\,(t' - t^{VI}) \ldots\ldots (t' - t^{(n)}) = \Pi'_1$$
$$(t'' - t^{IV})\,(t'' - t^{V})\,(t'' - t^{VI}) \ldots\ldots (t'' - t^{(n)}) = \Pi''_1$$
$$(t''' - t^{IV})\,(t''' - t^{V})\,(t''' - t^{VI}) \ldots\ldots (t''' - t^{(n)}) = \Pi'''_1$$

Then we have (3)

$$y' = \frac{\Omega'}{(t' - t'')\,(t' - t''')\,\Pi'_1} = \frac{1}{hk} \cdot \frac{\Omega'}{\Pi'_1}$$

$$y'' = \frac{\Omega''}{(t'' - t')\,(t'' - t''')\,\Pi''_1} = \frac{1}{h\,(h-k)} \cdot \frac{\Omega''}{\Pi''_1}$$

$$y''' = \frac{\Omega'''}{(t''' - t')\,(t''' - t'')\,\Pi'''_1} = \frac{1}{k\,(k-h)} \cdot \frac{\Omega'''}{\Pi'''_1}$$

If we add these three results, we obtain

$$y' + y'' + y''' = \frac{1}{hk} \cdot \frac{\Omega'}{\Pi'_1} + \frac{1}{h(h-k)} \cdot \frac{\Omega''}{\Pi''_1} + \frac{1}{k(k-h)} \cdot \frac{\Omega'''}{\Pi'''_1}$$

or, when we omit the infinitely small magnitude in this

last equation, and put $\Pi_1''' = \Pi_1'' = \Pi_1'$,

$$y' + y'' + y''' = \frac{1}{\Pi_1'}\left(\frac{\Omega'}{hk} + \frac{\Omega''}{h(h-k)} + \frac{\Omega'''}{k(k-h)}\right).$$

7. But by Taylor's Theorem

$$\Omega'' = \Omega' + \frac{d'\Omega}{dt'}\cdot\frac{h}{1} + \frac{d^2\Omega'}{dt'^2}\cdot\frac{h^2}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

$$\Omega''' = \Omega' + \frac{d'\Omega}{dt'}\cdot\frac{k}{1} + \frac{d^2\Omega'}{dt'^2}\cdot\frac{k^2}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{k^3}{1.2.3} + \&c.$$

If we substitute this sum in the expression for $y' + y'' + y'''$, and omit what ought to be left out, we then obtain

$$y' + y'' + y''' = \left(\frac{d^2\Omega'}{dt'^2}\cdot\frac{1}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{h+k}{1.2.3} + \&c.\right)\frac{1}{\Pi_1'}.$$

Now, if we put h and $k = o$, we get

$$y' + y'' + y''' = \frac{d^2\Omega'}{dt'^3}\cdot\frac{1}{1.2\Pi_1'}$$

8. In like manner, if four roots t', t'', t''', $t^{\prime v}$, of the equation II are equal to one another, when in the beginning we assume these roots as differing by an infinitely small magnitude, and $t'' = t' + h$, $t''' = t + k$, $t^{\prime v} = t' + l$, but after completing the calculation, we put h, k and $l = o$, we then find the following result:

$$y' + y'' + y''' + y^{\prime v} = \frac{d^3\Omega'}{dt'^3}\cdot\frac{1}{1.2.3\Pi_2'}.$$

when we put

$$(t' - t^{v})\,(t' - t^{v\prime})\,(t' - t^{v\prime\prime})\ldots\ldots(t' - t^{(n)}) = \Pi_2'$$

9. Hence we may perceive the law. Thus, if ν roots

$t', t'', t''', t^{IV}, \ldots\ldots t^{(\nu)}$ are equal to one another, we have

$$y'+y''+y''' \ldots\ldots y^{(\nu)} = \frac{d^{\nu-1}\Omega'}{dt'^{\nu-1}} \cdot \frac{1}{1 \cdot 2 \cdot 3 \ldots \nu-1 \cdot \Pi'}$$

when we put

$$(t'-t^{(\nu+1)})\,(t'-t^{(\nu+2)}) \ldots\ldots (t'-t^{(\pi)}) = \Pi'.$$

10. The expression Π' contains the roots $t^{\nu+1}$, $t^{\nu+2}$, $\ldots\ldots t^{(\pi)}$. Now, since it may happen that we know no other root of the equation II, except t', it remains to be shown, how we can determine this expression directly from the above equation.

11. By the assumed nature of the equation II, when we put

$$(t-t^{(\nu+1)})\,(t-t^{(\nu+2)})\,(t-t^{(\nu+3)}) \ldots\ldots (t-t^{(\pi)}) = \Pi$$

we have

$$(t-t')^{\nu}\Pi = t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + \&c.$$

If we differentiate this equation ν times successively with reference to t, and then substitute t' for t, we then obtain (foregoing §)

$$1 \cdot 2 \cdot 3 \ldots\ldots \nu \cdot \Pi' = \frac{d^{\nu}\,(\pi t'^{\pi} + Pt'^{\pi-1} + Qt'^{\pi-2} + \&c.)}{dt'^{\nu}}$$

or, when we actually differentiate it once

$$1 \cdot 2 \cdot 3 \ldots \nu \cdot \Pi' = \frac{d^{\nu-1}\,(\pi t'^{\pi-1} + (\pi-1)\,Pt'^{\pi-2} + (\pi-2)\,Qt'^{\pi-3} + \&c.}{dt'^{\nu-1}}$$

12. If we substitute the value of Π', which we derive from hence, in 9 we then obtain

$$y' + y'' + \dots + y^{\nu} = \frac{\nu d^{\nu-1}\Omega'}{d^{\nu-1}(\pi t'^{\pi-1} + (\pi-1)Pt'^{\pi-2} + \&c.)}$$

the differentials taken with reference to t'.

13. We have ∴ found the sum of the values corresponding to the equal values of y. But in like manner also, we may find the sum of their squares, cubes, and so on. To effect this, we only require in the equations in 3, § CXLII, for the function y to substitute its square y^2, its cube y^3, &c. Since by these means the magnitudes $z_0, z_1, z_2, \dots\dots z_{\pi-1}$, only undergo any change, nothing remains to be done, but to change the expression $\Omega = z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \dots\dots + U'z_0$ accordingly, and moreover to retain the formula just found for $y' + y'' + y''' + \dots\dots + y^{(\nu)}$. Having obtained these sums, we may likewise always find the equation, which has the values $y', y'', y''', \dots\dots y^{(\nu)}$ as roots, and this equation must necessarily be solved, if we wish to find the above values.

Remark. From what has been here said, we see the reason why it was said in 10, § CXXXVI, that it would be sufficient to solve the equation for the coefficient p, in order to find the other coefficients q, r, &c. immediately, and without the solution of any other equation. For since p, q, r, &c. are all homogeneous functions of x', x'', x''', &c. from the known numerical value of one of them, we may represent the numerical values of all the others by mere rational expressions; because the cases in which the denominators of these expressions vanish, belong to the exceptions, and can only occur in particular

equations, and not in general ones, of which we treated in the above-mentioned §.

SECTION CXLVII.

For the sake of the use which we might, perhaps, make of this, I shall now arrange the results found in the foregoing § together, and for the greater generality, instead of the function y itself, I shall assume any power of it $y^{\varkappa}$.

If we denote the symmetrical functions expressed by the coefficients of the given equation

$$\begin{array}{l} y'^{\varkappa} + \quad y''^{\varkappa} + \quad y'''^{\varkappa} + \ldots\ldots + \qquad (y^{(\pi)})^{\varkappa} \\ t'y'^{\varkappa} + \ t''y''^{\varkappa} + \ t'''y'''^{\varkappa} + \ldots\ldots + \ (t^{(\pi)})\ (y^{(\pi)})^{\varkappa} \\ t'^2y'^{\varkappa} + \ t''^2y''^{\varkappa} + \ t'''^2y'''^{\varkappa} + \ldots\ldots + \ (t^{(\pi)})^2\ (y^{(\pi)})^{\varkappa} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ t'^{\pi-1}y'^{\varkappa} + t''^{\pi-1}y''^{\varkappa} + t'''^{\pi-1}y'''^{\varkappa} + \ldots + (t^{(\pi)})^{\pi-1}(y^{(\pi)})^{\varkappa} \end{array}$$

in the order in which they succeed each other, by z_0, z_1, $z_2, \ldots\ldots z_{\pi-1}$, and put, for shortness' sake,

$$\pi t'^{\pi-1} + (\pi-1)Pt'^{\pi-2} + (\pi-2)\,Qt'^{\pi-3} + \&c. = \Phi'$$

$$z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots\ldots\ldots + U'z_0 = \Omega'$$

(in which $P' = t' + P$, $Q' = t'^2 + Pt' + Q$, $R' = t'^3 + Pt'^2 + Qt' + R$, &c.); we have for a simple root of the transformed equation for t,

$$y'^{\varkappa} = \frac{\Omega'}{\Phi'};$$

for a double root

$$y'^{\varkappa} + y''^{\varkappa} = \frac{2d\Omega'}{d\Phi'};$$

for a three-fold root

$$y'^{\varkappa} + y''^{\varkappa} + y'''^{\varkappa} = \frac{3d^2\Omega'}{d^2\Phi'};$$

for a four-fold root

$$y'^{\kappa} + y''^{\kappa} + y'''^{\kappa} + y'^{\mathrm{v}\kappa} = \frac{4d^3\Omega'}{d^3\Phi'};$$

and, in general, for an ν-fold root

$$y'^{\kappa} + y''^{\kappa} + y'''^{\kappa} + \ldots\ldots + (y^{(\nu)})^{\kappa} = \frac{\nu d^{\nu-1}\Omega'}{d^{\nu-1}\Phi'};$$

all the differentials taken in reference to t'.

By means of these formulæ, we may find the sums of powers of all those values of y, which belong to the complex root t'. Having now found these sums of powers, we may also, by § IX, find the equation on which they depend. I shall now elucidate what has been just advanced by an example.

Example. I shall assume, that in order to solve the equation

$$\text{I.} \quad x^4 - 3x^3 - 3x^2 + 11x - 6 = 0$$

we have transformed it into another

$$\text{II.} \quad t^6 - 9t^5 + 21t^4 + 9t^3 - 54t^2 + 32 = 0$$

when we put $t = x' + x''$. I shall further assume, that we are able to find a root of this last equation, and that we now wished from it to determine the value of the function $y = x'x''$.

Here we have the following corresponding values of t and y

$x' + x''$, $x' + x'''$, $x' + x^{\mathrm{iv}}$, $x'' + x'''$, $x'' + x^{\mathrm{iv}} + x''' + x^{\mathrm{iv}}$

$x'x''$, $x'x'''$, $x'x^{\mathrm{iv}}$, $x''x'''$, $x''x^{\mathrm{iv}}$, $x'''x^{\mathrm{iv}}$

and these give, when $\kappa = 1$,

$$z_0 = x'x'' + x'x''' + x'x^{\mathrm{iv}} + x''x''' + x''x^{\mathrm{iv}} + x'''x^{\mathrm{iv}}$$
$$= [1^2]$$

$$z_1=(x'+x'')\,x'x''+(x'+x''')\,x'x'''+(x+x'^{v})\,x'x'^{v}+\&c.$$
$$=[12]$$
$$z_2=(x'+x'')^2x'x''+(x'+x''')^2x'x'''+(x'x'^{v})^2x'x'^{v}+\&c.$$
$$=[13]+2[2^2]$$
$$z_3=(x'+x'')^3x'x''+(x'+x''')^3x'x'''+(x'+x'^{v})^3x'x'^{v}+\&c.$$
$$=[14]+3[23]$$
$$z_4=(x'+x'')^4x'x''+(x'x''')^4x'x'''+(x'+x'^{v})^4x'x'^{v}+\&c.$$
$$=[15]+4[24]+6[3^2]$$
$$z_5=(x'+x'')^5x'x''+(x'+x''')^5x'x'''+(x'+x'^{v})^5x'x'^{v}+\&c.$$
$$=[16]+5[25]+10[34]$$

If we take the numerical expressions from the annexed Tables, and then put for A, B, C, D, their values $3, -3, -11, -6$, we then find $z_0=-3$, $z_1=24$, $z_2=90$, $z_3=390$, $z_4=1542$, $z_5=6174$. If we substitute these values in the expression for Ω', we then obtain, since here $P'=t'-9$, $Q'=t'^2-9t'+21$; $R'=t'^3-9t'^2+21t'+9$, $S'=t'^4-9t'^3+21t'^2+9t'-54$, $T'=t'^5-9t'^4+21t'^3+9t'^2-54t'$, after the usual reduction:

$$\Omega'=z_5+P'z_4+Q'z_3+R'z_2+S'z_1+T'z_0$$
$$=-3t'^5+51t'^4-189t'^3+57t'^2+300t'$$

Also

$$\phi'=6t'^5-45t'^4+84t'^3+27t'^2-108t'$$

consequently

$$y'=\frac{-3t'^5+51t'^4-189t'^3+57t'^2+300t'}{6t'^5-45t'^4+84t'^3+27t'^2-108t'}$$

One root of the equation II, is $t=1$. If we substitute this root for t' in the value of y' here found, we get $y'=x'x''=-6$. Of the accuracy of this result we can convince ourselves by solving the two equations $x'+x''=1$, $x'x''=-6$; for by these means we obtain

3 and -2 for x' and x'', and these are actually two roots of the equation I.

Another root of the equation II, is $t=2$; and this root substituted for t' in the value of y', gives $y'=1$. But from $x'+x''=2$, and $x'x''=1$, we find $x'=x''=1$; whence it follows, that $x=1$ is also a root of the equation II, and that a double one.

But $t=4$ is also a root of the equation II. If we substitute this root for t' in the value of x', we find $y=\frac{o'}{o}$, which denotes that y' can be determined from t' in no other way than by an equation of the second degree. If, however, we differentiate Ω' and ϕ', we then find

$$d\Omega'=(-15t'^4+204t'^3-567t'^2+114t'+300)\,dt'$$
$$d\Phi'=(30t'^4-180t'^3+252t'^2+54t'-108)\,dt'$$

$\therefore$

$$y'+y''=\frac{2d\Omega'}{d\Phi'}=$$
$$2\,.\,\frac{-15t'^4+204t'^3+567t'^2+114t'+300}{30t'^4-180t'^3+252t'^2+54t'-108}.$$

If in this we put $t'=4$, we then get

$$y'+y''=6.$$

In order to determine y' and y'' singly, we must now find the value of $y'^2+y''^2$.

With this view, we put $\kappa=2$; we then have

$$z_0=x'^2x''^2+\&c.=[2^2]$$
$$z_1=(x'+x'')\,x'^2x''^2+\&c.=[23]$$
$$z_2=(x'+x'')^2x'^2x''^2+\&c.=[24]+2\,[3^2]$$
$$z_3=(x'+x'')^3x'^2x''^2+\&c.=[25]+3\,[34]$$
$$z_4=(x'+x'')^4x'^2x''^2+\&c.=[26]+4\,[35]+6\,[4^2]$$
$$z_5=(x'+x'')^5x'^2x''^2+\&c.=[27]+5\,[36]+10\,[45].$$

If we take the numerical expressions from the annexed Tables, and then put for A, B, C, D, their values 3, -3, -11, -6, we then find $z_0 = 63$, $z_1 = 102$, $z_2 = 336$, $z_3 = 1188$, $z_4 = 4668$, $z_5 = 18492$. If we substitute these values in the expression for Ω', and at the same time for P', Q', R', S', T', put the above expressions, after the requisite reduction, we find

$$\Omega' = z_5 + P'z_4 + Q'z_3 + R'z_2 + S'z_1 + T'z_0$$
$$= 63t'^5 - 465t'^4 + 741t'^3 + 873t'^2 - 1452t' - 1056$$

$\therefore$

$$d\Omega' = (315t'^4 - 1860t'^3 + 2223t'^2 + 1746t' - 1452)\,dt'$$

the values of Φ' and $d\Phi'$ remain the same as before. We obtain $\therefore$

$$y'^2 + y''^2 = 2 \cdot \frac{315t'^4 - 1860t'^3 + 2223t'^2 + 1746t' - 1452}{30t'^4 - 180t'^3 + 252t'^2 + 54t' - 108}$$

If in this we put $t' = 4$, we obtain

$$y'^2 + y''^2 = 18.$$

We have now $\therefore$ the two equations

$$y' + y'' = 6, \quad y'^2 + y''^2 = 18,$$

whence it follows, that the two values y', y'', depend on the quadratic equation

$$y^2 - 6y + 9 = 0$$

which contains the double root 3; and $\therefore$ $y' = y'' = 3$. That the result is correct, appears immediately, when we solve the two equations $x' + x'' = 4$, $x'x'' = 3$; for these give 1 and 3 for the values of x' and x'', which are actually two roots of the equation I. Besides, because here y' depends on an equation of the second degree, we may infer from hence, that $t = 4$ must be a double root of the equation II; which is also correct.

If we put $t = -1$, which is also a double root of the equation II, we find, when in the above expression for y', we put -1 for t', $y' = \frac{0}{0}$, as required. But if in the two expressions found for $y' + y''$ and $y'^2 + y''^2$, we put -1 for t', we then obtain

$$y' + y'' = -4,\ y'^2 + y''^2 = 8$$

and consequently the values of y', y'', depend on the equation

$$y^2 + 4y + 4 = 0$$

which has the double root $y = -2$. We have $\therefore$ $y' = y'' = -2$. But when we solve the two equations $x' + x'' = -1$, $x'x'' = -2$, we then obtain for x' and x'' the values 1 and -2, which are actually two roots of the equation I.

Besides that for y, as well as for $t = 4$, and for $t = -1$, we found such quadratic equations as have double roots, is merely accidental, and this will only be the case, when the equal values of y also correspond to the equal values of t.

SECTION CXLVIII.

Prob. Let t and y be any two functions of the roots of a given equation: required to find a general method by which, from the known value of one, to find the value of the other, however the functions are constituted.

Solution I. In order to solve the problem in its most general form, we shall assume, that both functions contain all the roots of the given equation. This supposition is

always allowable; for if one function does not contain all the roots at the same time, we then can, as was already observed in § XLIX, add those that are wanting with the coefficient o. Thus, if we had the function $x'x''x'''$, and the given equation were of the fifth degree, it would only be necessary, instead of these, to put $x'x''x''' + 0 \,.\, x'^{V} + 0 \,.\, x^{V}$.

2. The method in § CXLII for determining the numerical values of y from the numerical values of t assumed as known, in the case in which both these functions are symmetrical, may also be applied, when they are not so, by merely making the alterations which are requisite on this account. It was said in the above place, that, when $t', t'', t''', \ldots\ldots t^{(\pi)}$ denote the unequal values of forms of t, and $y', y'', y''', \ldots\ldots t^{(\pi)}$ the unequal values of forms of y, the function $t'^{\lambda}y' + t''^{\lambda}y'' + t'''^{\lambda}y''' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ is symmetrical, because the function $t^{\lambda}y$ can have no more unequal values of forms than those of which the former function is composed. This is correct, when the functions t, y, are no longer symmetrical, because they do not in this case change, or remain unchanged, at the same time.

3. But the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots. + (t^{(\pi)})^{\lambda}y^{(\pi)}$, in every imaginable state of the functions t, y, are assuredly always symmetrical, when $t', t'', t''', \ldots\ldots t^{(\pi)}$ and $y', y'', y''', \ldots\ldots y^{(\pi)}$ denote not only the unequal values of forms, but generally all the possible values, which arise from the transposition of the roots x', x'', x''', &c. whether equal or unequal. That in certain cases, and in

certain forms of the functions t, y, we often get a much less number of these values, is nothing to the purpose; because here we only are treating of the general method applicable to every case.

4. The method in § CLXIII for determining the numerical value of a function y from a single known numerical value of t, may in like manner be extended to functions which are not homogeneous, provided by t', t'', t''', ... $t^{(\pi)}$ and y', y'', y''', ... $y^{(\pi)}$, we merely denote all the possible values of forms of t and y, which arise from the transposition of the roots x', x'', x''', &c., and the transformed equation II be composed of all the values of forms of t, and not, as has always been the case hitherto, only of the unequal ones. This equation, however, will be found by the following method. I shall assume, that amongst all the π values of forms there are μ unequal ones, and that the equation for these last $t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + rt^{\mu-3} + \&c. = 0$ is already found. Further, if we put $\pi = \mu\nu$, then ν is necessarily a whole number, because by § LV, Corollary, μ is always a submultiple of π, and all the π values of forms, taken ν and ν together, will then be equal. The equation II, which, as is now required, is composed of all the values of forms t', t'', t''', ... $t^{(\pi)}$, is consequently no other than

$$(t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + rt^{\mu-3} + \&c.)^{\nu} = 0$$

and it may ∴ be obtained by solving this equation. If the values of forms t', t'', t''', ... $t^{(\pi)}$ be all different

from one another, then $\nu = 1$, and the equation II is the equation $t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + \&c. = 0$ itself.

5. With respect to the equation on which the numerical value of the function y depends, two cases must be distinguished; viz. 1st, the case in which the given equation is the most general one of its degree, and consequently whose coefficients are in no way combined; 2ndly, the case in which the coefficients are determinate numbers, or else have some relation to one another.

6. In the first case, the equation II can only contain roots which are all unequal, when the values of forms t', t'', t''' ...... $t^{(\pi)}$ are all different from one another; and if this be the case, as we have seen in the foregoing §, the numerical value of y may be expressed rationally by the numerical value of t. But if the above values of forms of t, consequently also the roots of the equation II, are equal, taken ν and ν together, then each of these roots is ν-fold, and consequently the numerical value of y (when all the particular relations between the functions t and y are first laid aside), necessarily depends on an equation of the νth degree, which may always be found (§ CXLVI); and this equation gives the ν values of y, which at the same time correspond to this root.

7. In the second case, on the other hand, it may happen, that this or that root t' of the equation II, besides the $\nu - 1$ equal values, which arise from the

identity of the values of forms, has also other equal ones, which have their bases in the particular property of the given equation itself, and consequently in a case of this kind the numerical value of y, which corresponds to the root t', must necessarily be given by an equation of a higher degree than the *vth.*

8. Hitherto we have not noticed, in the general inquiries respecting the dependence of the numerical values of the functions t and y, the particular nature of these functions; it is now time to consider this. We have already seen, in the preceding §, that, when the above functions are symmetrical, the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ becomes symmetrical, when for $t', t'', t''', \ldots\ldots t^{(\pi)}$ we merely take the unequal values of forms; by which means not only the calculation is essentially shortened, but likewise in the case, in which the transformed equation for t has equal roots, the numerical values of y, which correspond to these equal roots, are expressed by lower equations than we should have obtained if we had introduced all the values of forms of t. But a similar abbreviation may generally be used, when the functions t and y are such, that when the nature of one of them is expressed by the equation

$$A' = A'' = A''' = \ldots = A^{(k)} = A^{(k+1)} = \ldots\ldots = A^{(p)}$$

between the p types $A', A'', A''', \ldots\ldots A^{(k)}, A^{(k+1)}, \ldots\ldots A^{(p)}$, the nature of the other is determined by the equation

$$A' = A'' = A''' = \ldots = A^{(k)}$$

merely between the k types $A', A'', A''', \ldots A^{(k)}$. For

if we try to find all the unequal types, which a function contains, whose nature is expressed by the type-equation $A' = A'' = A''' = \ldots\ldots = A^{(i)}$, and then find all the values of forms of the function t and y corresponding to these unequal types; consequently, when t', t'', t''', $\ldots\ldots t^{(\pi)}$ and y', y'', y''', $\ldots\ldots y^{(\pi)}$ denote these values, the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ will necessarily be symmetrical, because there is no value of forms of $t^{\lambda}y$, which is not included amongst those of which this aggregate is composed.

9. As for the formation of the transformed equation for t in the assumed property of the functions t, y, we must distinguish the two cases, where t or y is that function, whose nature is determined by the equation $A' = A'' = A''' = \ldots\ldots = A^{(i)}$. If the first be supposed, then the values of forms t', t'', t''',......$t^{(\pi)}$, are all different from one another, and the equation II, which is composed of these values of forms, is actually, as in the case of homogeneous functions, only the result of the unequal values of forms. But if the second supposition be taken, then amongst the values of forms t', t'', t''',...$t^{(\pi)}$ there are several equal ones; and when we put the numbers of the unequal ones amongst these, consequently the number of the unequal values of forms which that function has whose nature is determined by the type-equation $A' = A'' = A''' = \ldots = A^{(\rho)}$, equal to μ, the number π is a multiple of the number μ. If $\therefore$ we put $\pi = \mu\nu$, and assume that $t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + \&c. = 0$ is the equation, which is merely composed of the unequal values of forms of t, then the equation II, which is composed of

the values of forms t', t'', t''',......$t^{(\pi)}$, is no other than the developement of the equation

$$(t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + \&c.)^{\nu} = o.$$

10. Further, since to each root t of the equation II there are ν corresponding values of y, consequently the numerical value of y depends necessarily on an equation of the νth degree. If the functions t and y be homogeneous, then $\nu = 1$, and consequently this value depends only on an equation of the first degree, as required. But all this only obtains so long as the given equations are general ones; for in particular equations it might certainly happen, as was already observed, that the equation for y were of a higher degree.

11. Besides the relations given in 8, between the functions t, y, there are numberless others, in which the calculation may, in like manner, be simplified. Such a simplification as this is always practicable, when in all the values of forms of t, which arise from all the possible transpositions of the roots x', x'', x''', &c., such as t', t'', t''',......$t^{(\mu)}$ may be omitted, which are all either different, or the periods of the different values occur more than once, and at the same time are such, that the function $t'^{\lambda}y' + t''^{\lambda}y'' + t'''^{\lambda}y''' + \ldots\ldots + (t^{(\mu)})^{\lambda}y^{(\mu)}$ is symmetrical.

12. Although there are cases where the calculation may be simplified, when, instead of all the values of forms of the function t, we only use those which possess

the properties just mentioned, yet for the determination of the value of y from the value of t, there is no further disadvantage arising from it (with the exception of the calculations being extremely prolix). It may indeed be objected, that in this case the equation for y rises to a higher degree than is necessary, and that it may happen, that we cannot solve an equation of this kind, notwithstanding, perhaps, that in the calculation properly arranged, we arrive at a solvible equation. But since in this case amongst the roots of the former equation, there must be more than one which are equal, and in the sequel it will be shown, that an equation of this kind may always be reduced to another, which only contains the unequal roots, consequently in the present case the lowest rational equation for y, $\therefore$ this objection is removed of itself.

SECTION CXLIX.

Prob. Let t and y be two functions of the roots of a general equation of any degree: required to give the degree of the equation, by which the numerical value of y is determined from the known numerical value of t.

Solution. In the function t perform all the transpositions of the roots x', x'', x''', &c. for which its value of form remains unchanged; in the function y perform the same transpositions as in t. Let ν be the number of the unequal values of forms of y, which we obtain by these means; then the equation between y and t, with reference to y, is of the νth degree. For since the equal

values of forms of t have only a single numerical value, the ν unequal values of forms, on the other hand, ν different numerical values, consequently υ numerical values of y belong to a single numerical value of t; $\therefore$ the former can be determined from the last in no other way than by an equation of the νth degree.

EXAMPLE I. With respect to the general equation of the fourth degree, let $t = f:\ (x')\ (x'')\ (x''')\ ('^{v})$, $y = \phi:\ (x')\ (x'')\ (x''')\ (x'^{v})$, and let the nature of these functions be expressed by the type-equations

$$f:\ (x')\ (x'')\ (x''')\ (x'^{v}) = f:\ (x'')\ (x')\ (x''')\ (x'^{v}) =$$
$$f:\ (x')\ (x'')\ (x'^{v})\ (x''') = f:\ (x''')\ (x'^{v})\ (x')\ (x'')$$
$$\phi:\ (x')\ (x'')\ (x''')\ (x'^{v}) = \phi:\ (x'')\ (x')\ (x''')\ (x'^{v})$$
$$= \phi:\ (x')\ (x'')\ (x'^{v})\ (x''')$$

Now, if we try to find the equal values of forms of t (§ LV), and then perform the same transpositions in y, we then obtain the following corresponding values of t and y:

$f:\ (x')\ (x'')\ (x''')\ (x'^{v})$	$\phi:\ (x')\ (x'')\ (x''')\ (x'^{v})$
$f:\ (x')\ (x'')\ (x'^{v})\ (x''')$	$\phi:\ (x')\ (x'')\ (x'^{v})\ (x''')$
$f:\ (x'')\ (x')\ (x''')\ (x'^{v})$	$\phi:\ (x'')\ (x')\ (x''')\ (x'^{v})$
$f:\ (x'')\ (x')\ (x'^{v})\ (x''')$	$\phi:\ (x'')\ (x')\ (x'^{v})\ (x''')$
$f:\ (x''')\ (x'^{v})\ (x')\ (x'')$	$\phi:\ (x''')\ (x'^{v})\ (x')\ (x'')$
$f:\ (x'^{v})\ (x''')\ (x')\ (x'')$	$\phi:\ (x'^{v})\ (x''')\ (x')\ (x'')$
$f:\ (x''')\ (x'^{v})\ (x'')\ (x')$	$\phi:\ (x''')\ (x'^{v})\ (x'')\ (x')$
$f:\ (x'^{v})\ (x''')\ (x'')\ (x')$	$\phi:\ (x'^{v})\ (x''')\ (x'')\ (x')$

Of the eight values of y, which we have here found, the four first, as well as the four last, on account of the sup-

posed nature of this function, are equal to one another; consequently two values of y belong to a single value of t. Therefore the equation, which gives y in terms of t, is of the second degree.

To the numberless functions of the assumed nature, the following ones belong, viz.: $t = x'x'' + x'''x^{IV}$, $y = x'x'' = x'x'' + 0\,x''' + 0\,x^{IV}$, or $y = x' + x'' = x' + x'' + 0\,x''' + 0\,x^{IV}$. Consequently, if the numerical value of $x'x'' + x'''x^{IV}$ be known, we may find from it both the numerical value of $x'x''$, and that of $x' + x''$, by the solution of an equation of the second degree, which agrees with § XLI, where we merely had to solve an equation of the second degree.

Example II. For any general equation, let $t = x'x''x''' + x^{IV}$, $y = x' - x'' = x' - x'' + 0\,(x''' + x^{IV})$. Now, in order to find the degree of the lowest rational equation, by which y may be determined from t, proceed as follows:

Equal values of forms of t	Corresponding values of forms of y
$x'x''x''' + x^{IV}$	$x' - x'' + 0\,(x''' + x^{IV})$
$x'x'''x'' + x^{IV}$	$x' - x''' + 0\,(x'' + x^{IV})$
$x''x'x''' + x^{IV}$	$x'' - x' + 0\,(x''' + x^{IV})$
$x''x'''x' + x^{IV}$	$x'' - x''' + 0\,(x' + x^{IV})$
$x'''x'x'' + x^{IV}$	$x''' - x' + 0\,(x'' + x^{IV})$
$x'''x''x' + x^{IV}$	$x''' - x'' + 0\,(x' + x^{IV})$

Consequently six different values of y belong to a single value of t, viz.: $x' - x''$, $x' - x'''$, $x'' - x'$, $x'' - x'''$,

$x''' - x'$, $x''' - x''$; and $\therefore$ y can only be determined from t by an equation of the sixth degree, when its coefficients are rational functions of t. Besides, since the values of forms of y, taken two and two, are equal, this equation $\therefore$ only contains even powers of y.

SECTION CL.

PROB. Let t, y, be any two functions of the roots $x'x''x'''$ &c. of a general equation: required an operation to find the lowest equation, by which the numerical value of y may be determined from the numerical value of t, under the condition that the unequal values of forms of t only are made use of.

Solution. Find, as in the foregoing §, the equal values of forms of t, and the corresponding values of forms of y, and from these last take away the unequal ones; let them be $y', y'', y''' \ldots\ldots\ldots y^{(\nu)}$: then the required equation will be of the νth degree, and it has the above values for roots. Let

$$y^{\nu} + py^{\nu-1} + qy^{\nu-2} + ry^{\nu-3} + \&c. = 0$$

be this equation; then $p = y' + y'' + y''' + \&c.$ $q = y'y'' + y'y''' + y''y''' + \&c.$ $r = y'y''y''' + \&c.$ &c. Consequently, since the functions p, q, r, &c. with reference to $y', y'', y''', \ldots\ldots y^{(\nu)}$, are symmetrical, then in those transpositions of the roots x', x'', x''', &c. for which the function t continues unchanged, they in like manner undergo no change. If $\therefore$ we denote the unequal values of forms of t, by $t', t'', t''', \ldots\ldots\ldots t^{(\pi)}$, then

$t'^{\lambda}p'$, $t''^{\lambda}p''$, $t'''^{\lambda}p'''$, ... $(t^{(\pi)})^{\lambda}p^{(\pi)}$, are all the possible unequal values of forms of $t^{\lambda}p$, and in the same way $t'^{\lambda}q'$, $t''^{\lambda}q''$, $t'''^{\lambda}q'''$, ... $(t^{(\pi)})^{\lambda}q^{(\pi)}$ are all the possible values of forms of $t^{\lambda}q$, and so on. But if this be symmetrical with reference to x', x'', x''', &c. .... then also are the functions $t'^{\lambda}p' + t''^{\lambda}p'' + t'''^{\lambda}p''' + \ldots$ $\ldots + (t^{(\pi)})^{\lambda}p^{(\pi)}$, $t'^{\lambda}q' + t''^{\lambda}q'' + t'''^{\lambda}q''' + \ldots\ldots\ldots\ldots +$ $(t^{(\pi)})^{\lambda}q^{(\pi)}$ necessarily symmetrical with reference to x', x'', x''', &c., and consequently the unequal values of forms t', t'', t''', ... $t^{(\pi)}$ are sufficient for the determination of p, q, r, &c. Therefore the operation given in § CXLVI may be applied immediately, and, without any alteration to the coefficients p, q, r, &c. Thus, if we wish to determine p, we find, in the first place, the transformed equation for the function t according to the third chapter; it is

$$t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + \&c. = 0.$$

Having found this, we immediately get

$$p = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'_{\pi-3} + \ldots\ldots + U'z_0}{\pi t^{\pi-1} + \pi-1 \,.\, Pt^{\pi-2} + \pi-2 \,.\, Qt^{\pi-3} + \&c.}$$

in which $P' = t + P$, $Q' = t^2 + Pt + Q$, &c., and the symbol of the form z_k denotes the numerical value of the symmetrical function $t'^{k}p' + t''^{k}p'' + t'''^{k}p''' + \ldots$ $\ldots\ldots + (t^{(\pi)})^{k}p^{(\pi)}$. For the coefficients q, r, &c. the same equation and the same expression obtain, with this exception, that by z_k we must understand the numerical values of the functions $t'^{k}q' + t''^{k}q'' + t'''^{k}q''' + \ldots\ldots$ $\ldots + (t^{(\pi)})^{k}q^{(\pi)}$, $t'^{k}r' + t''^{k}r'' + t'''^{k}r''' + \ldots\ldots\ldots +$ $(t^{(\pi)})^{k}r^{(\pi)}$, &c.

SECTION CLI.

So long as x', x'', x''', &c. may be considered as the roots of a general equation $x^n + Ax^{n-1} + Bx^{n-2} + \&c. = 0$, consequently of one whose coefficients A, B, C, &c. are undetermined, we shall always find rational functions of t for the coefficients p, q, r, &c. But if these roots relate to a particular equation, then it may happen, according to the nature of the function t, that the common denominator $\pi t^{\pi-1} + \overline{\pi - 1} \,.\, Pt^{\pi-2} + \overline{\pi - 2} \,.\, Qt^{\pi-3} + \&c.$ in the expressions for p, q, r, &c. is equal to 0, and that it even continues equal to 0, when it is differentiated more than once. We now assume, that we must differentiate it $\mu - 1$ times before the denominator ceases to vanish, consequently it follows, from § CLXVI, that the coefficients p, q, r, &c. depend upon the same number of equations of the μth degree

$$p^\mu + a'p^{\mu-1} + b'p^{\mu-2} + c'p^{\mu-3} + \&c. = 0$$
$$q^\mu + a''q^{\mu-1} + b''q^{\mu-2} + c''q^{\mu-3} + \&c. = 0$$
$$r^\mu + a'''r^{\mu-1} + b'''r^{\mu-2} + c'''r^{\mu-3} + \&c. = 0$$
$$\&c.$$

which may always be found by the method there given, and in which the coefficients a', b', c', &c. a'', b'', c'', &c. a''', b''', c''', &c. are all rational functions of t.

All that has been said in this chapter respecting the function y_2 may also be applied to the function x. Thus, if we wish to determine a root, say x', from the known value of a function $t = f: (x')\,(x'')\,(x''')\ldots(x^n)$, nothing further is necessary, than to put $y = x'$, and to proceed besides in the way already pointed out.

We now perceive the reason why we are not able, from

the known value of a symmetrical function of the roots of an equation, whatever the nature of this function may be, to determine these roots. For since a function of this kind, in all the transpositions of the roots, always retains the same value, ∴ it must necessarily give all the roots at once; and, however we begin it, we shall consequently always again get an equation, which is not different from the given one.

VIII.—A GENERAL METHOD FOR THE SOLUTION OF EQUATIONS OF ALL DEGREES.

SECTION CLII.

IN § CXLI we have seen that the requisites for the general solution of equations may be reduced to two; viz. first, to find such functions of the roots, by means of which the equation, into which we have transformed the given one, is adapted to the solution; and secondly, to determine the roots from the known value of the assumed functions. The second requisite we have handled in the foregoing chapter; the first, together with its application to the general solution of equations, will form the subject of this chapter.

In order to render the notation more easy, and the inspection more convenient, I shall henceforth omit in the types the letter x, together with the superfluous brackets, and for the dashes substitute numbers; thus: f: (12345 ......n) instead of f: $(x') (x'') (x''') (x^{\prime v})...(x^{(n)})$, and f: (342651) instead of f: $(x''') (x^{\prime v}) (x'') (x^{v\prime}) (x^{v}) (x')$.

SECTION CLIII.

Rule. When from the period of n types A_1, A_2, A_3, A_4......A_μ....A_ν......A_n, which may be derived from the equation

$$f:\ (123456\ldots\ldots \overline{n-1 n}) = f:\ (234567\ldots\ldots n1)$$

we take away any two A_μ, A_ν, and find all the possible types, which may be derived from the transformation-rule $A_\mu = A_\nu$; then we shall get a period, which, in the case where $\nu - \mu$ and ν are prime numbers to each other, consists of all the n types $A_1, A_2, A_3, \ldots\ldots A_n$; on the other hand, when $\nu - \mu$ and n have a common measure m, this period only consists of $\frac{n}{m}$ of these types. The types, which we successively obtain by deduction, succeed each other in the following order:

$$A_\mu,\ A_\nu,\ A_{2\nu-\mu},\ A_{3\nu-2\mu},\ A_{4\nu-3\mu},\ \&c.$$

so that the dashes μ, ν, $2\nu-\mu$, $3\nu-2\mu$, $4\nu-3\mu$, &c. form an arithmetical progression, with the difference $\nu-\mu$, when from all the terms of this progression, which exceed the number n, we omit this number as often as possible.

Thus the equation

$$f:\ (12345678) = f:\ (23456781)$$

gives the period

$$\begin{array}{l} A_1 \ldots\ldots\ldots\ldots f:\ (12345678) \\ A_2 \ldots\ldots\ldots\ldots f:\ (23456781) \\ A_3 \ldots\ldots\ldots\ldots f:\ (34567812) \\ A_4 \ldots\ldots\ldots\ldots f:\ (45678123) \\ A_5 \ldots\ldots\ldots\ldots f:\ (56781234) \\ A_6 \ldots\ldots\ldots\ldots f:\ (67812345) \\ A_7 \ldots\ldots\ldots\ldots f:\ (78123456) \\ A_8 \ldots\ldots\ldots\ldots f:\ (81234567) \end{array}$$

If we equate every two of these types, we obtain

For the equation	The period
$A_1 = A_3$	$A_1,\ A_3,\ A_5,\ A_7,$
$A_1 = A_4$	$A_1,\ A_4,\ A_7,\ A_2,\ A_5,\ A_8,\ A_3,\ A_6$
$A_1 = A_5$	$A_1,\ A_5$
$A_1 = A_6$	$A_1,\ A_6,\ A_3,\ A_8,\ A_5,\ A_2,\ A_7,\ A_4$
$A_1 = A_7$	$A_1,\ A_7,\ A_5,\ A_3$
$A_1 = A_8$	$A_1,\ A_8,\ A_7,\ A_6,\ A_5,\ A_4,\ A_3,\ A_2$
$A_2 = A_3$	$A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8,\ A_1$
$A_2 = A_4$	$A_2,\ A_4,\ A_6,\ A_8$
$A_2 = A_5$	$A_2,\ A_5,\ A_8,\ A_3,\ A_6,\ A_1,\ A_4,\ A_7$
&c.	&c.

The reason of this is easily found, and depends on the properties of numbers.

Corollary I. The rule is also correct, when, instead of the type-equation $A_\mu = A_\nu$ we take the type-equation $A_\nu = A_\mu$, if in the progression which is taken away, viz. $\nu,\ \mu,\ 2\mu - \nu,\ 3\mu - 2\nu,\ 4\mu - 3\nu,\ 5\mu - 4\nu$, &c., when we come to a negative term or o, we add the number n so many times, till it becomes positive. Thus we have

For the equation	The period
$A_2 = A_1$	$A_2,\ A_1,\ A_8,\ A_7,\ A_6,\ A_5,\ A_4,\ A_3$
$A_3 = A_1$	$A_3,\ A_1,\ A_7,\ A_5$
$A_4 = A_1$	$A_4,\ A_1,\ A_6,\ A_3,\ A_8,\ A_5,\ A_2,\ A_7$
$A_5 = A_1$	$A_5,\ A_1$
$A_6 = A_1$	$A_6,\ A_1,\ A_4,\ A_7,\ A_2,\ A_5,\ A_8,\ A_3$
$A_7 = A_1$	$A_7,\ A_1,\ A_3,\ A_5$
$A_8 = A_1$	$A_8,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7$
&c.	&c.

Corollary II. If $\therefore$ n be a prime number, we then always obtain the same period again, whichever two of the types A_1, A_2, A_3, ... A_n we put equal to one-another.

SECTION CLIV.

Transpositions of the kind, which the equation $A_u = A_v$, or $A_v = A_\mu$, gives in the preceding §, are called recurring transpositions, and the periods which are obtained from them, recurring periods.

The characteristic feature of transpositions of this kind consists in this, that each root is removed one place forwards or backwards, and in the first case the last takes the place of the first; but in the second case, the first takes the place of the last, so that there is a kind of circular motion; as if, for instance, a number of persons stand in a circle, having their backs to each other, and all walking at the same time either backwards or forwards.

The transpositions are called recurring ones, when only some of the roots move in the manner just mentioned, but the remaining ones retain their places. Thus the equation f: (12345678) = f: (34512678) only gives recurring transpositions of the first five roots. The law of the preceding § is also true for this, when we take for n merely the number of the roots to be transposed, and the remaining ones are considered, as though they did not exist.

SECTION CLV.

Rule. If the equation $x^n + Ax^{n-1} + Bx^{n-2} +$ &c. $=0$, by the introduction of a function $t = f$: $(12345\ldots n)$,

be transformed into an equation of two terms $t^n - K = 0$: then the roots of this last equation $t', t'', t''', \ldots\ldots t^{(n)}$ are always the numerical values of such values of forms of the function t, as, taken together, form a period.

Proof. The roots of the equation $t^n - K = 0$ may always, as may be seen from the fifth chapter, be expressed by $t', at', a^2t', a^3t', \ldots\ldots a^{n-1}t'$, when a denotes a primitive root of the equation $t^n - 1 = 0$. Therefore $t'' = at', t''' = at'', t^{IV} = at''', \ldots\ldots t^{(n)} = at^{(n-1)}$, $t' = at^{(n)}$. Now, let $A_1, A_2, A_3, A_4, \ldots\ldots A_n$, denote the values of forms, which correspond to the roots t', t'', t''', $\ldots\ldots t^{(n)}$, then also must $A_2 = aA_1$, $A_3 = aA_2$, $A_4 = aA_3$, $\ldots\ldots A_n = aA_{n-1}$, $A_1 = aA_n$; and since every such equation $A\nu = aA_{\nu-1}$, independently of the particular values which we may assign to the roots x', x'', x''', &c. must be true, it will also remain true, when in the two parts of this equation we transpose the above roots in any way, provided it be the same, because this is precisely the same as when the values of these roots are changed. If ∴ we assume, that in $A_{\nu-1}$, we have so transposed the roots x', x'', x''', &c. that it becomes A_ν, and that by the same transposition A_ν is transformed into any other value of form A_π; then we have also $A_\pi = aA_\nu$. But if also $A_{\nu+1} = aA_\nu$, consequently $A_{\nu+1} = A_\pi$. Hence it follows that $A_{\nu+1}$ is generated by the same transposition from A_ν, as A_ν is from $A_{\nu-1}$; ∴ A_2 is produced from A_1, as A_3 is from A_2, as A_4 from A_3; and so on; lastly, as A_n is from A_{n-1}, and A_1 from A_n. Since ∴ all the values of forms $A_1, A_2, A_3, A_4, \ldots\ldots\ldots A_n$ are deduced from one another by the same transposition-rule, and

from the last A_n, the first A_1 is again obtained; $\therefore$ it follows, that these n values of forms constitute a period.

SECTION CLVI.

PROB. Amongst all the possible functions of the roots x', x'', of the general equation of the second degree $x^2 - Ax + B = 0$, find that one which is fit for its solution; under the supposition that we know not how to solve any other equations, than those of the first degree, and those of the form $t^2 - K = 0$.

Solution 1. Let $t = f\colon (12)$ be that function, which is fit for the solution of the given equation. Since it has two values, viz. $f\colon (12)$, $f\colon (21)$, then the equation for t, taken generally, is of the second degree. If we only wished it to be of the first degree, then must $f\colon (12) = f\colon (21)$; but then $f\colon (12)$ would be symmetrical; and the roots x', x'', could be determined from the known value of t only by the solution of the given equation itself (§ CXLIX). There now remains nothing further than to assume that the two values of forms $f\colon (12)$, $f\colon (21)$, are the roots of an equation of the form $t^2 - K = 0$, because it was assumed that we know not how to solve an equation of the second degree of any other form but this. That they may be so, appears from hence, that they form a period (preceding §).

2. Since $K = -t't''$, when t', t'' denote the two roots of the equation $t^2 - K = 0$; then also $K = - f\colon (12) \times f\colon (21)$; and since this product remains

the same when we substitute x' for x'', $\therefore K$ is a symmetrical function of these roots. Consequently this magnitude may be expressed rationally by the coefficients of the given equation.

3. Since $f:(12)$, $f:(21)$ are the roots of the equation $t^2 - K = o$, therefore $f:(12) = -f:(21)$ and this is the only condition which we have to fulfil. Having once found the numerical value of $t = f:(12)$, then also the roots x', x'', may be determined without the solution of any other equation, because the values of forms $f:(12)$, $f:(21)$ are different (§ CXLIV).

4. This condition, however, is evidently sufficient, when we put $f:(12) = \phi:(12) - \phi:(21)$, where it is allowable, for $\phi:(12)$ to assume every arbitrary function of x', x'' which is not symmetrical. For from $f:(12) = \phi:(12) - \phi:(21)$ we obtain by the substitution of x' for x'', $f:(21) = \phi:(21) - \phi:(12)$: consequently $f:(12) = -f:(21)$, as was required.

5. Hence it follows, that all functions of the form $\phi:(12) - \phi:(21)$ are fit for the solution of the given equation.

SECTION CLVII.

Prob. Solve actually the general equation of the second degree $x^2 - Ax + B = o$.

Solution 1. We have seen in the foregoing §, that all functions of the form $t = \phi:(12) - \phi:(21)$ are fit for

the solution. Amongst the infinite number of functions which we can assume for $\phi:$ (12), the root x is the most simple. Put $\therefore \phi:$ $(12)=x'$, then $t=\phi:(12)-\phi:(21)$ $=x'-x''$. But the equation $t^2-K=0$ gives $K=t^2=$ $(x'-x'')^2=x'^2+x''^2-2x'x''=[2]-2[1^2]=A^2-4B$; the transformed equation consequently is

$$t^2-(A^2-4B)=0$$

and this gives $t=\pm\sqrt{(A^2-4B)}$. We $\therefore$ have the two equations

$$x'+x''=A$$
$$x'-x''=\pm\sqrt{(A^2-4B)}$$

and hence $x'=\dfrac{A\pm\sqrt{(A^2-4B)}}{2}$, $x''=\dfrac{A\mp\sqrt{(A^2-4B)}}{2}$, as was required.

SECTION CLVIII.

Prob. Find the functions which are fit for the solution of the general equation of the third degree

$$x^3-Ax^2+Bx-C=0$$

under the supposition, that we know not how to solve any other equations than those of the first and second degrees, and those of the form $t^3-K=0$.

Solution 1. Let $t=f:(123)$ represent all those functions which are fit for the solution of the given equation. Since the roots x', x'', x''', admit of six transpositions, consequently the function t contains six values; and these are

$$f:(123),\quad f:(231),\quad f:(312)$$
$$f:(213),\quad f:(132),\quad f:(321)$$

Consequently, taken generally, the equation for t is of the sixth degree.

2. The six values of forms in 1 are arranged in recurring periods. Thus, in the first horizontal row, we have the recurring period $f:$ (123), and in the second $f:$ (213). If we assume, that the three values of forms of the first period are the roots of the equation of two terms, viz. $t^3 - K = o$, then K is the product of these three roots, and $\therefore = f: (123) \times f: (231) \times f: (312)$. But this product, as may be easily seen, is such, that in all the recurring transpositions of the three roots x', x'', x''', it undergoes no change: for if we perform these transpositions, we then obtain the period

$$\begin{array}{l} f: (123) \times f: (231) \times f: (312) \\ f: (231) \times f: (312) \times f: (123) \\ f: (312) \times f: (123) \times f: (231) \end{array}$$

and these three values of forms of K are evidently not different from one another. The six values of forms of K, which arise from all the transpositions of the roots x', x'', x''', are $\therefore$ equal, taken three and three, and consequently this function has no more than two different values; and these are

$$\begin{array}{l} f: (123) \times f: (231) \times f: (312) \\ f: (213) \times f: (132) \times f: (321) \end{array}$$

of which one can be derived from the other merely by putting the roots x', x'' for each other.

3. Since $\therefore K$ has no more than two different values, consequently this function depends on an equation of no higher degree than the second, and the roots of this equa-

tion are these two values. But these values admit of a more simple form; for since $K = t^3$, and $t = f\!:\ (123)$, then also $K = \big(f\!:\ (123)\big)^3$; and since the second value, as we have already seen in 2, is obtained from the first, merely by putting the roots x', x'' for each other, consequently $\big(f\!:\ (213)\big)^3$ is the second value of K.

4. Let

$$K^2 - pK + q = 0$$

be the equation, on which the function K depends, ∴ p the sum, and q the product, of the two values of K. Consequently

$$p = \big(f\!:\ (123)\big)^3 + \big(f\!:\ (213)\big)^3$$
$$q = \big(f\!:\ (123)\big)^3 \times \big(f\!:\ (213)\big)^3$$

and these functions p, q are such, that in all the transpositions of the roots x', x'', x''', they suffer no change. Since p and q are symmetrical functions of the roots x', x'', x''', they may always ∴ be expressed rationally by the coefficients of the given equation.

5. We have consequently reduced the transformed equation for t, which originally was of the sixth degree, to two equations

$$t^3 - K = 0$$
$$K^2 - pK + q = 0$$

and we are always able to determine the coefficients p, q, from A, B, C, when the function $f\!:\ (123)$ is known. Having once determined the coefficients p, q, we then obtain, by the solution of the second equation, the two values of K, and if these be successively substituted in the first equation, by its solution we obtain the six values of t.

6. Since all the values of t are different from one another, we may always determine (which is known from the foregoing chapter) the values of the roots x', x'', x''', immediately from the values of the function t already found, and that without the solution of any other equation, however constituted this function may be.

7. It now only remains to determine the function $t = f: (123)$ in such a way, that the three values of forms $f: (123)$, $f: (231)$, $f: (312)$, may be the roots of an equation of the form $t^3 - K = o$. If this be the case, then these three values must have such a relation to each other, that

$$f: (123) = \alpha f: (231) = \alpha^2 f: (312).$$

In order to perform this, we assume any other arbitrary function $\phi: (123)$, and put

$$f: (123) = A\phi: (123) + B\phi: (231) + C\phi: (312)$$

in which A, B, C, denote coefficients hitherto unknown. From this equation, by a proportional transposition of the roots, we obtain

$$f: (231) = A\phi: (231) + B\phi: (312) + C\phi: (123)$$
$$f: (312) = A\phi: (312) + B\phi: (123) + C\phi: (231)$$

and when we substitute these values in the foregoing proportional equation, we get

$$\begin{aligned} & A\phi: (123) + B\phi: (231) + C\phi: (312) \\ = {} & \alpha\left(A\phi: (231) + B\phi: (312) + C\phi: (123)\right) \\ = {} & \alpha^2\left(A\phi: (312) + B\phi: (123) + C\phi: (231)\right) \end{aligned}$$

If we equate the coefficients of these values of forms, we obtain, for the determination of A, B, C, the following equations:

$$A = \alpha C = \alpha^2 B$$
$$B = \alpha A = \alpha^2 C$$
$$C = \alpha B = \alpha^2 A.$$

Since $\alpha^3 = 1$, the first gives $B = \alpha A$, $C = \alpha^2 A$, and these values verify also the second and third. The coefficient A remains undetermined, and we may ∴ put it equal to 1. Consequently

$$f: (123) = \phi: (123) + \alpha\phi: (231) + \alpha^2\phi: (312).$$

8. We can, as was observed already, for $\phi: (123)$ assume every arbitrary function; yet, for another reason, those which undergo no change in the recurring transpositions of all the three roots, cannot be used. For in the case where $\phi: (123)$ is a function of this kind, we have $\phi: (123) = \phi: (231) = \phi: (312)$, and consequently $f: (123) = (1 + \alpha + \alpha^2)\,\phi: (123) = 0$, because $1 + \alpha + \alpha^2 = 0$. This restriction ∴ might with good reason be omitted, since it is a necessary consequence.

SECTION CLIX.

Prob. Required to solve actually the general equation of the third degree $x^3 - Ax^2 + Bx - C = 0$.

Solution 1. In the foregoing § we saw, that all functions of the form $\phi: (123) + \alpha\phi: (231) + \alpha^2\phi: (312)$ are fit for the solution of equations of the third degree. Consequently there are numberless ways in which these equations may be solved. The most simple supposition

is $\phi:\ (123) = x'$; then $\phi:\ (231) = x''$, $\phi:\ (312) = x'''$; $\therefore$

$$f:\ (123) = x' + \alpha x'' + \alpha^2 x'''.$$

2. Hence we obtain

$$\big(f:\ (123)\big)^3 = [3] + 6[1^3] +$$
$$3\alpha(x''x'^2 + x'x''^2 + x'''x''^2) + 3\alpha^2(x'''x'^2 + x'x''^2 + x''x'''^2)$$

and when in this we substitute x' for x'', we get

$$\big(f:\ (213)\big)^3 = [3] + 6[1^3] +$$
$$3\alpha(x'x''^2 + x''x'''^2 + x'''x'^2) + 3\alpha^2(x'''x''^2 + x''x'^2 + x'x'''^2)$$

or when, for shortness' sake, we put $[3] + 6[1^3] = P$, $x''x'^2 + x'x'''^2 + x'''x''^2 = Q$, $x'''x'^2 + x'x''^2 + x''x'''^2) = R$,

$$\big(f:\ (123)^3\big) = P + 3\alpha Q + 3\alpha^2 R$$
$$\big(f:\ (213)\big)^3 = P + 3\alpha R + 3\alpha^2 Q.$$

3. Hence, by 4 of the foregoing §, we further obtain

$$p = \big(f:\ (123)\big)^3 + \big(f:\ (213)\big)^3$$
$$= 2P + 3(\alpha + \alpha^2)(Q + R)$$

or, since $\alpha + \alpha^2 = -1$, and $Q + R = [12]$,

$$p = 2[3] + 12[1^3] - 3[12]$$

and when for the numerical expressions we put their values taken from the annexed Tables,

$$p = 2A^3 - 9AB + 27C.$$

4. Further, by the foregoing §

$$q = \big(f:\ (123)\big)^3 \times \big(f:\ (213)\big)^3 =$$
$$(P + 3\alpha Q + 3\alpha^2 R)(P + 3\alpha R + 3\alpha^2 Q)$$
$$= P\big(P + 3(\alpha+\alpha^2)(Q+R)\big) + 9(\alpha+\alpha^2)QR + 9(Q^2 + R^2)$$

or, since $\alpha + \alpha^2 = -1$, $Q + R = [12]$, $QR = [3^2] + 3[2^3] + [1^2 4]$, $Q^2 + R^2 = [24] + 2[123]$

$$q = \big([3] + 6[1^3]\,([3] + 6[1^3] - 3[12]\big)$$
$$+ 9\big([24] + 2[123] - [3^2] - 3[2^3] - [1^2 4]\big)$$

or when for the numerical expressions we put their values,

$$q = A^6 - 9A^4B + 27A^2B^2 - 27B^3$$
$$= (A^2 - 3B)^3.$$

5. Consequently, the two equations in 5 of the foregoing § are

$$t^3 - K = 0$$
$$K^2 - (2A^3 - 9AB + 27C)\,K + (A^2 - 3B)^3 = 0.$$

6. Let K', K'' be the two values of K, and t', t'', the corresponding values of t, then

$$t' = f:\ (123) = x' + ax'' + a^2x''' = \sqrt[3]{K'}$$
$$t'' = f:\ (213) = x'' + ax' + a^2x''' = \sqrt[3]{K''}.$$

We have ∴ for the determination of x', x'', x''', the three equations

$$x' + x'' + x''' = A$$
$$x' + ax'' + a^2x''' = \sqrt[3]{K'}$$
$$x'' + ax' + a^2x''' = \sqrt[3]{K''}.$$

If we multiply the third equation by a^2, and then add it to the two first, after dividing by 3, we then get, since $1 + a + a^2 = 0$,

$$x' = \frac{A + \sqrt[3]{K'} + a^2\sqrt[3]{K''}}{3}.$$

If we multiply the second by a^2, and then add it to the other two, we then get

$$x'' = \frac{A + \alpha^2\sqrt[3]{K'} + \sqrt[3]{K''}}{3}.$$

Lastly, if we multiply the two last equations by α, and then add them to the first, we obtain

$$x''' = \frac{A + \alpha(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}.$$

7. But since each of the two irrational magnitudes $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, has three values, for instance, the first the values $\alpha\sqrt[3]{K'}$, $\alpha^2\sqrt[3]{K'}$, $\alpha^3\sqrt[3]{K'}$, and the second the values $\alpha\sqrt[3]{K''}$, $\alpha^2\sqrt[3]{K''}$, $\alpha^3\sqrt[3]{K''}$, we must first determine which are to be taken. I assert, in the first place, that the two roots $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, must always be combined with one and the same power of α. For let

$$x' + \alpha x'' + \alpha^2 x''' = \alpha^\nu\sqrt[3]{K'}$$

in which the exponent ν may either be one, two, or three. If in this equation we put the roots x' and x'' for one another, we then obtain

$$x'' + \alpha x' + \alpha^2 x''' = \alpha^\nu\sqrt[3]{K''}$$

because in this case K' is transformed into K'', ∴ also $\sqrt[3]{K'}$ into $\sqrt[3]{K''}$. Hence it follows, that in the two last of the three equations, consequently also in the results derived from them, the roots $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, must always be combined with the same power of α. It only remains now to determine the exponent ν.

8. With this view, if we put in the values of x', x'', x''', found in 6, $\alpha^{\nu}\sqrt[3]{K'}$, $\alpha^{\nu}\sqrt[3]{K''}$ for $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, respectively, then

$$x' = \frac{A + \alpha^{\nu}\sqrt[3]{K'} + \alpha^{\nu+2\nu}\sqrt[3]{K''}}{3}$$

$$x'' = \frac{A + \alpha^{\nu+2\nu}\sqrt[3]{K'} + \alpha^{\nu}\sqrt[3]{K''}}{3}$$

$$x''' = \frac{A + \alpha^{\nu+1}(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}$$

Now, since these three roots must also be found, when for α the other primitive root α^2 is substituted, then also

$$x''' = \frac{A + \alpha^{2\nu+2}(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}.$$

must be a root. But since this one is not to be found amongst the three here given, consequently no other assumption is allowable, except this one, that $\alpha^{\nu+1} = \alpha^3 = 1$; $\therefore$ $\nu = 2$. Consequently $\alpha^2\sqrt[3]{K'}$, $\alpha^2\sqrt[3]{K''}$, must be substituted for $\sqrt[3]{K'}$, $\sqrt[3]{K''}$. If we actually do this, we find the three following equations:

$$(A + \sqrt[3]{K'} + \sqrt[3]{K''}) : 3$$
$$(A + \alpha\sqrt[3]{K'} + \alpha^2\sqrt[3]{K''}) : 3$$
$$(A + \alpha^2\sqrt[3]{K'} + \alpha\sqrt[3]{K''}) : 3$$

in which it is only necessary for K' and K'' to substitute the two roots of the second equation in 5.

9. If, for the sake of greater simplicity, we put $A = 0$, then this equation is transformed into

$$K^2 - 27CK - 27B^3 = 0$$

and the two values of K are

$$27[\tfrac{1}{2}C \pm \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)].$$

If we substitute these values in the three roots in 3, we obtain

$$\sqrt[3]{[\tfrac{1}{2}C + \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)]} + \sqrt[3]{[\tfrac{1}{2}C - \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)]}$$

$$\alpha\sqrt[3]{[\tfrac{1}{2}C + \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)]} + \alpha^2\sqrt[3]{[\tfrac{1}{2}C - \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)]}$$

$$\alpha^2\sqrt[3]{[\tfrac{1}{2}C + \surd(\tfrac{1}{8}C^2 + \tfrac{1}{27}B^3)]} + \alpha\sqrt[3]{[\tfrac{1}{2}C - \surd(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)]}$$

which agrees with Cardan's formula.

SECTION CLX.

Prob. Find the functions which are fit for the solution of the general equation of the fourth degree, viz.

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

under the supposition that we only know how to solve equations of lower degrees, and those of the form $t^4 - K = 0$.

Solution 1. Arrange the twenty-four values of forms of $f:$ (1234) in recurring transpositions, under and opposite each other; (the symbolical function and the brackets are omitted for shortness' sake)

1 2 3	4	2 3 4 1	3 4 1 2	4 1 2 3
2 3 1	4	3 1 4 2	1 4 2 3	4 2 3 1
3 1 2	4	1 2 4 3	2 4 3 1	4 3 1 2
2 1 3	4	1 3 4 2	3 4 2 1	4 2 1 3
1 3 2	4	3 2 4 1	2 4 1 3	4 1 3 2
3 2 1	4	2 1 4 3	1 4 3 2	4 3 2 1.

Thus, in the first vertical column, we first put $f:$ (1234) with its recurring transpositions of the three first roots,

this gives the recurring period $f:(1234)$, $f:(2314)$, $f:(3124)$. Then we in like manner put $f:(2134)$ with its recurring transpositions of the three first roots, and we obtain the period $f:(2134)$, $f:(1324)$, $f:(3214)$. From the value of form $f:(1234)$ by a recurring transposition of all the four roots, we further derive the values of forms $f:(2341)$, $f:(3412)$, $f:(4123)$, and place them near $f:(1234)$ in a horizontal row; we do the same with the remaining five values of forms in the first vertical column, so that in each horizontal row there is a recurring period.

2. Since the four values of forms in the first horizontal row form a period, they may ∴ be the roots of an equation of the form

$$t^4 - K = 0$$

(§ 155). Now, since $-K$ is the product of all the four roots, we have

$$-K = f:(1234) \times f:(2341) \times f:(3412) \times f:(4123)$$

and this product is such, that in all the transpositions of the roots we can obtain no more than the following six different values:

$$f:(1234) \times f:(2341) \times f:(3412) \times f:(4123)$$
$$f:(2314) \times f:(3142) \times f:(1423) \times f:(4231)$$
$$f:(3124) \times f:(1243) \times f:(2431) \times f:(4312)$$
$$f:(2134) \times f:(1342) \times f:(3421) \times f:(4213)$$
$$f:(1324) \times f:(3241) \times f:(2413) \times f:(4132)$$
$$f:(3214) \times f:(2143) \times f:(1432) \times f:(4321)$$

which arise merely from the transposition of the three roots x', x'', x'''.

3. From the equation $t^4 - K = o$, we obtain $K = t^4 = (f: (1234))^4$. Consequently also $(f: (1234))^4$ must be such a function, that in the recurring transpositions of all the four roots it remains the same, and consequently has no more different values than those which arise from the transposition of the roots x', x'', x'''. Therefore the six values of K can also be expressed thus:

$$(f: (1234))^4,\ (f: (2314))^4,\ (f: (3124))^4$$
$$(f: (2134))^4,\ (f: (1324))^4,\ (f: (3214))^4.$$

4. Since the function K has still six different values, consequently it necessarily depends on an equation of the sixth degree. If this equation be solvible, then it must admit of being reduced to such equations, whose solution is assumed to be known. I shall .·. assume, that the three functions $(f: (1234))^4$, $(f: (2314))$,^{4} $(f: (3124))^4$, which arise from the recurring transposition of the three roots x', x'', x''', are the roots of an equation of the third degree

$$\text{I. } K^3 - pK^2 + qK - r = o$$

consequently the coefficients p, q, r, are no longer rational, because otherwise K can have no more than three values. They must .·. depend on certain equations, which we shall now seek.

5. Since $(f: (1234))^4$, $(f: (2314))^4$, $(f: (3124))^4$, are the roots of the equation I, then

$$p = (f: (1234))^4 + (f: (2314))^4 + (f: (3124))^4$$

$$q = (f:\ (1234))^4 \times (f:\ (2314))^4$$
$$+ (f:\ (1234))^4 \times (f:\ (3124))^4$$
$$+ (f:\ (2314))^4 \times (f:\ (3124))^4$$
$$r = (f:\ (1234))^4 \times (f:\ (2314))^4 \times (f:\ (3124))^4.$$

The functions p, q, r, are evidently such, that in the recurring transpositions of the three roots x', x'', x''' they undergo no change. But in the recurring transpositions of all the four roots $x', x'', x''', x^{\prime v}$, they in like manner suffer no change, because the functions $(f:\ (1234))^4$, $(f:\ (2314))^4$, $(f:\ (3124))^4$, remain the same after this operation (3).

6. Consequently the functions p, q, r, can have no more than two different values, viz. those which arise merely from the transposition of the roots x', x''. If ∴ we put, for shortness' sake, $p = f':\ (1234)$, then p has no more than the two values $f':\ (1234), f':\ (2134)$. Let these two values be the roots of the following equation of the second degree:

$$p^2 - p'p + q' = 0$$

then

$$p' = f':\ (1234)\ \times f':\ (2134)$$
$$q' = f':\ (1234)\ \times f':\ (2134).$$

The functions p', q', are ∴ such, that when x' is substituted for x'', they remain unchanged. But since in the recurring transpositions of the three roots x', x'', x''', as also in the recurring transpositions of all the four roots, they also suffer no change; they ∴, are necessarily symmetrical, and consequently admit of being expressed rationally by the coefficients of the given equation.

What has been here said of p, may, in like manner, be said of q and r. Consequently these coefficients also depend on equations of the second degree with rational coefficients.

7. The function $t = f:(1234)$.˙. depends on the equation of two terms of the fourth degree, viz.

$$t^4 - K = o$$

and the coefficient K depends again on the equation of the third degree

$$K^3 - pK^2 + qK - r = o$$

whose coefficients p, q, r, are represented by three equations of the second degree

$$\begin{aligned} p^2 - p' p + q' &= o \\ q^2 - p'_1 q + q'_1 &= o \\ r^2 - p'_2 r + q'_2 &= o \end{aligned}$$

whose coefficients p', q', p'_1, q'_1, p'_2, q'_2, are all rational functions of the coefficients A, B, C, D.

8. It only remains now to determine the function $f:(1234)$ in such a way, that the values of forms $f:(1234)$, $f:(2341)$, $f:(3412)$, $f:(4123)$ may be the roots of an equation of the form $t^4 - K = o$. If this be the case, then they must have such a relation to one another, that

$f:(1234) = \alpha f:(2341) = \alpha^2 f:(3412) = \alpha^3 f:(4123)$.

Now, in order to perform this, we put in a way similar to that in 7, § CLVIII,

$$f:(1234) =$$
$$A\phi:(1234) + B\phi:(2341) + C\phi:(3412) + D\phi:(4123)$$

and derive from hence the values of $f:(2341)$, $f:(3412)$,

$f: (4123)$. If we substitute now these values in the foregoing proportional equation, we then get

$$\begin{aligned} & A\varphi: (1234) + B\varphi: (2341) + C\varphi: (3412) + D\varphi: (4123) \\ = {} & \alpha\,(A\varphi: (2341) + B\varphi: (3412) + C\varphi: (4123) + D\varphi: (1234)) \\ = {} & \alpha^2\,(A\varphi: (3412) + B\varphi: (4123) + C\varphi: (1234) + D\varphi: (2341)) \\ = {} & \alpha^3\,(A\varphi: (4123) + B\varphi: (1234) + C\varphi: (2341) + D\varphi: (3412)) \end{aligned}$$

and when we put the coefficients of these values of forms equal to one another

$$\begin{aligned} A &= \alpha D = \alpha^2 C = \alpha^3 B \\ B &= \alpha A = \alpha^2 D = \alpha^3 C \\ C &= \alpha B = \alpha^2 A = \alpha^3 D \\ D &= \alpha C = \alpha^2 B = \alpha^3 A. \end{aligned}$$

The first equation gives $B=\alpha A$, $C=\alpha^2 A$, $D=\alpha^3 A$; and these values verify also the second, third, and fourth equations. We have consequently

$$f: (1234) =$$
$$\phi: (1234) + \alpha\phi: (2341) + \alpha^2\phi: (3412) + \alpha^3\phi: (4123).$$

SECTION CLXI.

Prob. Solve actually the general equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

under the same conditions as those of the foregoing problem.

Solution 1. We have seen in the foregoing §, that all functions of the form $\phi: (1234) + \alpha\phi: (2341) + \alpha^2\phi: (3412) + \alpha^3\phi: (4123)$ are adapted to the solution. If, for the sake of greater perspicuity, we put

$\phi:\ (1234) = x'$, then

$$f:\ (1234) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v}$$

or, when we briefly substitute for α one of the primitive roots of the equation $x^4 - 1 = o$, say $+ \sqrt{-1}$,

$$f:\ (1234) = x' - x''' + (x'' - x^{\prime v}) \sqrt{-1}.$$

Hence we obtain

$$f:\ (2314) = x'' - x' + (x''' - x^{\prime v}) \sqrt{-1}$$
$$f:\ (3124) = x''' - x'' + (x' - x^{\prime v}) \sqrt{-1}.$$

2. By 5 of the foregoing §, we ∴ have

$$p = \left(x'\ x''' + (x''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 + \left(x''\text{-}x' + (x'''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 + \left(x'''\text{-}x'' + (x'\text{-}x^{\prime v}) \sqrt{-1}\right)^4$$

$$q = \left(x'\text{-}x''' + (x''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 \times \left(x''\text{-}x' + (x'''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 + \left(x'\text{-}x''' + (x''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 \times \left(x'''\text{-}x'' + (x'\text{-}x^{\prime v}) \sqrt{-1}\right)^4 + \left(x''\text{-}x' + (x'''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 \times \left(x'''\text{-}x'' + (x'\text{-}x^{\prime v}) \sqrt{-1}\right)^4$$

$$r = \left(x'\text{-}x''' + (x''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 \times \left(x''\text{-}x' + (x'''\text{-}x^{\prime v}) \sqrt{-1}\right)^4 \times \left(x'''\text{-}x'' + (x'\text{-}x^{\prime v}) \sqrt{-1}\right)^4$$

The functions p, q, r, are evidently such, that each of them can only have a single value which is different, viz. that which arises from the one here given, when we substitute x'' for x', and, vice versâ, x' for x''. Consequently each of these functions depends on an equation of the second degree, which may be found by the method already well known from the foregoing §. As the subject contains no difficulty, and the calculation is rather diffuse, I shall dwell no longer upon it.

3. In the preceding chapter we have seen, that in two homogeneous functions the value of one may always be determined from the value of the other by a rational

expression, so long as we have to do with general equations. Consequently also the values of q, r may be found immediately from the known value of p. Now, since p has two values, consequently the magnitudes p, q, r may be determined in two different ways. Every such determination gives an equation

$$K^3 - pK^2 + qK - r = 0$$

and we $\therefore$ obtain generally six values of K. If we put $K = f' : (1234)$, then the six values of forms, which correspond to these numerical values, are those which arise from the transposition of the first three roots (§ CLX, 3); $\therefore f' : (1234)$, $f' : (2314)$, $f' : (3124)$, $f' : (2134)$, $f' : (1324)$, $f' : (3214)$, of which the three first correspond to the three roots of one equation for K, and the three last to the three roots of the other.

4. Let K', K'', K''', be the three roots, which correspond to the values of forms $f' : (1234)$, $f' : (2314)$, $f' : (3124)$. If we substitute these values of K in the equation $t^4 - K = 0$, we then obtain for t three values $\sqrt[4]{K'}$, $\sqrt[4]{K''}$, $\sqrt[4]{K'''}$, and to these $\therefore$ the values of forms $f : (1234)$, $f : (2314)$, $f : (3124)$ correspond. Now, since $f : (1234) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v}$, we then have, including the equation $x' + x'' + x''' + x^{\prime v} = A$, the four following equations:

$$\begin{aligned}
x' + x'' + x''' + x^{\prime v} &= A \\
x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} &= \sqrt[4]{K'} \\
x'' + \alpha x''' + \alpha^2 x' + \alpha^3 x^{\prime v} &= \sqrt[4]{K''} \\
x''' + \alpha x' + \alpha^2 x'' + \alpha^3 x^{\prime v} &= \sqrt[4]{K'''}.
\end{aligned}$$

5. If we multiply the three last by α, and then add them to the first, we obtain, after dividing by four

$$x^{IV} = \frac{A + \alpha(\sqrt[4]{K'} + \sqrt[4]{K''} + \sqrt[4]{K'''})}{4}$$

and the remaining roots x', x'', x''', are all of the form $aA + b\sqrt[4]{K'} + c\sqrt[4]{K''} + d\sqrt[4]{K'''}$, in which a, b, c, d, denote certain functions of α, which are different for each of these roots.

6. It may now be shown in a similar way with that in 7 § CLIX, that the roots $\sqrt[4]{K'}$, $\sqrt[4]{K''}$, $\sqrt[4]{K'''}$, must be combined with the same power of α. For if we put

$$x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} = \alpha^{\nu}\sqrt[4]{K'}$$

then also must

$$x'' + \alpha x''' + \alpha^2 x' + \alpha^3 x^{IV} = \alpha^{\nu}\sqrt[4]{K''}$$

and

$$x''' + \alpha x' + \alpha^2 x'' + \alpha^3 x^{IV} = \alpha^{\nu}\sqrt[4]{K'''}$$

because K' in the first transposition is transformed into K'', and in the second into K''', but α^{ν} remains the same. Now I assert, that necessarily $\nu = 3$, since otherwise α would not vanish from the value of x^{V}, and consequently amongst the roots x', x'', x''', there must be another of the form of the root x^{IV}, because for α we could also have substituted the other primitive root of the equation $x^4 - 1 = 0$. We are certain ∴, that

$$x = \frac{A + \sqrt[4]{K'} + \sqrt[4]{K''} + \sqrt[4]{K'''}}{4}$$

is a root of the given equation; and we could also have

found the other roots, if we had given ourselves the trouble to solve the four equations in 4.

SECTION CLXII.

Prob. Find functions, which are fit for the solution of the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

under the supposition, that we know not how to solve any other equation but those of lower degrees, and those of the form $t^5 - K = 0$.

Solution 1. Arrange the 120 values of forms of the function $t = f : (12345)$ in recurring periods, as follows: (symbolical functions and brackets are omitted)

1 2 3 4 5	2 3 4 5 1	3 4 5 1 2	4 5 1 2 3	5 1 2 3 4
2 3 1 4 5	3 1 4 5 2	1 4 5 2 3	4 5 2 3 1	5 2 3 1 4
3 1 2 4 5	1 2 4 5 3	2 4 5 3 1	4 5 3 1 2	5 3 1 2 4
2 1 3 4 5	1 3 4 5 2	3 4 5 2 1	4 5 2 1 3	5 2 1 3 4
1 3 2 4 5	3 2 4 5 1	2 4 5 1 3	4 5 1 3 2	5 1 3 2 4
3 2 1 4 5	2 1 4 5 3	1 4 5 3 2	4 5 3 2 1	5 3 2 1 4
2 3 4 1 5	3 4 1 5 2	4 1 5 2 3	1 5 2 3 4	5 2 3 4 1
3 1 4 2 5	1 4 2 5 3	4 2 5 3 1	2 5 3 1 4	5 3 1 4 2
1 2 4 3 5	2 4 3 5 1	4 3 5 1 2	3 5 1 2 4	5 1 2 4 3
....	....	....	....	....
....	....	....	....	....
....	....	....	....	....
4 3 2 1 5	3 2 1 5 4	2 1 5 4 3	1 5 4 3 2	5 4 3 2 1

Thus in the first vertical column we find the 24 transpositions of the four first roots arranged under one another,

in the same way as they were in 1, § CLX. The four following columns contain the recurring transpositions of all the five roots, and in such a way, that in each horizontal row there is a period.

2. According to this arrangement, the 120 values of forms of the function $f\colon (12345)$ may $\therefore$ be generated in the following way. From the two values of forms, $f\colon (12345)$, $f\colon (21345)$, which form a period of recurring transpositions of the two first roots, we derive, by recurring transpositions of the three first roots, the six values of forms, $f\colon (12345)$, $f\colon (23145)$, $f\colon (31245)$, $f\colon (21345)$, $f\colon (13245)$, $f\colon (32145)$. From these, by recurring transpositions of the first four roots, we get the 24 values of forms which are contained in the first vertical column in 1; and lastly, from these again, by recurring transpositions of all the five roots, we derive all the 120 values of forms.

3. Let the five values of forms $f\colon (12345)$, $f\colon (23451)$, $f\colon (34512)$, $f\colon (45123)$, $f\colon (51234)$, be the roots of the equation of two terms

$$t^5 - K = 0$$

then K is their product, consequently

$$K = f\colon (12345) \times f\colon (23451) \times f\colon (34512) \\ \times f\colon (45123) \times f\colon (51234).$$

If, for the sake of brevity, we put $K = f'\colon (12345)$, then $f\colon (12345)$ is a function such, that in all the recurring transpositions of the five roots, it remains the same, because in each such transposition, one of the five factors, of

which it is composed, merely changes place with another. Consequently all the values of K include 24 times five equal values, and $\therefore$ this function can contain no more than 24 unequal values, and they are those which correspond to the 24 transpositions in the first vertical column in 1, and consequently those which arise exclusively from the transposition of the four roots, x', x'', x''', x'^{v}.

4. The equation for t, which, taken generally, is of the 120th degree, is consequently already reduced, by the introduction of the function K, to an equation of the 24th degree. Each root of this last equation gives five values of t, viz. $\sqrt[5]{K}$, $\alpha\sqrt[5]{K}$, $\alpha^2\sqrt[5]{K}$, $\alpha^3\sqrt[5]{K}$, $\alpha^4\sqrt[5]{K}$, and $\therefore$ all the 24 roots together give all the 120 values of t.

5. Since $K = t^5$ and $t = f: (12345)$, then also $K = f': (12345) = f: (12345)^5$. The 24 roots of the equation for K are $\therefore$ no other than the results of the transpositions of the four first roots in $\big(f: (12345)\big)^5$. We must now endeavour to reduce this equation.

6. With this view I shall assume, that the four values of forms $f': (12345)$, $f': (23415)$, $f': (34125)$, $f': (41235)$, which together constitute a period of recurring transpositions of the four first roots, are the roots of an equation of the fourth degree

$$K \; - pK^3 + qK^2 - rK + s = 0;$$

then the coefficients p, q, r, s, are symmetrical functions of these four values of forms, and, consequently, in each recurring transposition of the roots x', x'', x''' x'^{v}, they remain

the same, because in each such transposition, one of these values merely changes place with another. Therefore they can contain no more unequal values than those which arise from the transposition of the roots x', x'', x'''; consequently six values. Therefore the coefficients p, q, r, s, depend on equations of the sixth degree only.

7. Since p, q, r, s, are homogeneous functions, because they all change only when the roots x', x'', x''', are transposed, we are $\therefore$ always enabled, from the known value of one of these coefficients, say p, to find directly the corresponding values of the remaining ones q, r, s, (§ CXLIII). It is consequently quite sufficient to solve the equation for p. Moreover, the six corresponding values of p, q, r, s, give six such equations as those in 6; and since each of these equations gives four values of K, consequently all the six equations together give the 24 values of K.

8. If we put $p = f'' : (12345)$, then $f'' : (12345)$, $f'' : (23145)$, $f'' : (31245)$, $f'' : (21345)$, $f'' : (13245)$, $f'' : (32145)$, are the six unequal values of forms of p, which form two periods of recurring transpositions of the three roots x', x'', x'''. I shall now assume that the three values of forms of the first period $f'' : (12345)$, $f'' : (23145)$, $f'' : (31245)$, are the roots of an equation of the third degree

$$p^3 - p'p^2 + q'p - r' = 0;$$

$\therefore p'$, q', r', are symmetrical functions of these three values, and consequently in each recurring transposition of the three roots x', x'', x''', they remain the same. They

$\therefore$ can have no more than two different values, viz. those which arise merely from the substitution of x' for x''. Besides, if p' be found, then also q', r' may be found immediately, because these three functions are homogeneous.

9. Let $p' = f''' : (12345)$, then $f''' : (12345)$, $f''' : (21345)$ are the only two unequal values of forms of this function. If $\therefore$ we assume, that they are the roots of the equation

$$p'^2 - p''p' + q'' = 0$$

then p'', q'' are symmetrical functions of the roots x', x'', x''', $x^{\prime v}$, x^{v}, and may consequently be expressed rationally by the coefficients A, B, C, D, E, of the given equation.

10. We have now $\therefore$ reduced the equation for t of the 120th degree to the following equations:

$$\begin{aligned}
&\text{I.} \quad t^3 - K = 0 \\
&\text{II.} \quad K^4 - pK^3 + qK^2 - rK + s = 0 \\
&\text{III.} \quad p^3 - p'p^2 + q'p - r' = 0 \\
&\text{IV.} \quad p'^2 - p''p' + q'' = 0.
\end{aligned}$$

Having found the equation IV, we obtain from it two values of p'. If we substitute one of these values of p' in the equation III, and for q', r', the corresponding values, we then obtain, by the solution of this equation, three values of p. Lastly, if we substitute one of these values in the equation II, and for q, r, s, their corresponding values, we then obtain four values of K, and from one of these values that of t.

11. We now wish to inquire, how the function t must be constituted, in order that the five values of forms $f:(12345)$, $f:(23451)$, $f:(34512)$, $f:(45123)$, $f:(51234)$, may be the roots of an equation of the form $t^5 - K = o$; for this was the supposition with which we set out. If this condition be fulfilled, we then must have:

$$f:(12345) = \alpha f:(23451) = \alpha^2 f:(34512)$$
$$= \alpha^3 f:(45123) = \alpha^4 f:(51234).$$

Of this kind, however, are all the functions of the form

$$\phi:(12345) + \alpha\phi:(23451) + \alpha^2\phi:(34512)$$
$$+ \alpha^3\phi:(45123) + \alpha^4\phi:(51234).$$

Consequently all functions of this form are fit for the solution of an equation of the fifth degree, under the supposition that we are able, from the known value of this function, to determine the roots x', x'', x''', x^{IV}, x^{V}.

12. But I affirm, that this last supposition is always correct, whatever function we may assume for $\phi:(12345)$. For since t, in every recurring transposition of all the roots, changes its value, ∴ it can have at most only twenty-four equal values, viz. those which are in the first vertical column in t. Now, since amongst these values there is not a single one which has x^{V} in the first place, consequently, these can only at the same time give the roots x', x'', x''', x^{IV}, but the root x^{V} would always admit of being determined rationally from t. Therefore, equations of the fifth degree may be solved in an infinite number of ways, and we shall see in the sequel, that this is generally the case with all equations.

13. Since $f' : (12345) = \big(f : (12345)\big)^5$, and $p = f' : (12345) + f' : (23415) + f' : (34125) + f' : (41235)$, then also, since we have put $p = f'' : (12345)$,

$$f'' : (12345) = \big(f : (12345)\big)^5 + \big(f : (23415)\big)^5 + \big(f : (34125)\big)^5 + \big(f : (41235)\big)^5.$$

Further, since $p' = f'' : (12345) + f'' : (23145) + f'' : (31245) = f''' : (12345)$, then also, when the requisite transpositions of the roots are made,

$$\begin{aligned} f''' : (12345) = {} & \big(f : (12345)\big)^5 + \big(f : (23415)\big)^5 + \\ & \big(f : (34125)\big)^5 + \big(f : (41235)\big)^5 + \\ & \big(f : (23145)\big)^5 + \big(f : (31425)\big)^5 + \\ & \big(f : (14235)\big)^5 + \big(f : (42315)\big)^5 + \\ & \big(f : (31245)\big)^5 + \big(f : (12435)\big)^5 + \\ & \big(f : (24315)\big)^5 + \big(f : (43125)\big)^5. \end{aligned}$$

If in this we substitute x' for x'', we obtain

$$\begin{aligned} f''' : (21345) = {} & \big(f : (21345)\big)^5 + \big(f : (13425)\big)^5 + \\ & \big(f : (34215)\big)^5 + \big(f : (42135)\big)^5 + \\ & \big(f : (13245)\big)^5 + \big(f : (32415)\big)^5 + \\ & \big(f : (24135)\big)^5 + \big(f : (41325)\big)^5 + \\ & \big(f : (32145)\big)^5 + \big(f : (21435)\big)^5 + \\ & \big(f : (14325)\big)^5 + \big(f : (13425)\big)^5. \end{aligned}$$

Hence we see, that the two functions $f''' : (12345)$ and $f'' : (21345)$, taken together, give all the possible values of $\big(f : (12345)\big)^5$, which arise from the transposition of the four first roots, consequently all the unequal values of this function. Therefore, since $p'' = f''' : (12345) + f''' : (21345)$, p'' is a symmetrical function of x', x'', x''', x^{iv}, x^{v}, which is obtained immediately from the function $\big(f : (12345)\big)^5$, when we take the sum of all

the values of this function, which arise from the transposition of the four roots x', x'', x''', x'^{v}.

14. Further, because $q''=f''':(12345)\times f''':(21345)$, we then immediately obtain the function q'', by taking the product of the above two values for $f''':$ (12345) and $f''':$ (21345). Besides, it is evident, that both in p'' and q'' the root α must vanish, because otherwise t would have more than 120 values.

SECTION CLXIII.

Prob. Solve actually the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

Solution 1. Since for $\phi:$ (12345) in the foregoing §, we can assume any arbitrary function, in order ∴ to simplify the calculation, I shall assume for it the root x, and put $\phi:(12345) = x'$. Then $\phi:(23451) = x''$, $\phi:(34512)=x'''$, $\phi:(45123)=x'^{v}$, $\phi:(51234)=x^{v}$; hence

$$t = f:(12345) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v} + \alpha^4 x^{v}$$

and ∴

$$\big(f:(12345)\big)^5 = (x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v} + \alpha^4 x^{v})^5$$

to which expression we can also give, as in § CXL, the form

$$(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x'^{v} + \alpha^5 x^{v})^5.$$

2. If we solve this expression according to the powers

of α, it has, for the reasons given in 12, § CXXXIX, the following form:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{\mathrm{IV}}\alpha^3 + \xi^{\mathrm{V}}\alpha^4$$

and then

$$\xi' = [5] + 120\,[1^5] +$$

$$+20\left\{\begin{array}{c} x'^3x''x^{\mathrm{V}} + x'^3x'''x^{\mathrm{IV}} + x'x''^3x''' + x''^3x^{\mathrm{IV}}x^{\mathrm{V}} + \\ x'x'''^3x^{\mathrm{V}} + x''x'''^3x^{\mathrm{IV}} + x'x''x^{\mathrm{IV}3} + x'''x^{\mathrm{IV}3}x^{\mathrm{V}} \\ + x'x^{\mathrm{IV}}x^{\mathrm{V}3} + x''x'''x^{\mathrm{V}3} \end{array}\right\}$$

$$+30\left\{\begin{array}{c} x'x''^2x^{\mathrm{V}2} + x''x'''^2x^{\mathrm{IV}2} + x'^2x''x'''^2 + x''x^{\mathrm{IV}2}x^{\mathrm{V}2} + \\ x'^2x'''x^{\mathrm{V}2} + x''^2x'''x^{\mathrm{IV}2} + x'^2x''^2x^{\mathrm{IV}} + x'''^2x^{\mathrm{IV}}x^{\mathrm{V}2} \\ + x'^2x^{\mathrm{IV}2}x^{\mathrm{V}} + x'^2x'''^2x^{\mathrm{V}} \end{array}\right\}$$

$$\xi'' =$$

$$5\,(x'^4x'' + x''^4x''' + x'''^4x^{\mathrm{IV}} + x'x^{\mathrm{V}4} + x^{\mathrm{IV}4}x^{\mathrm{V}})$$

$$+10\,(x'^3x^{\mathrm{IV}2} + x'^2x''^3 + x''^3x^{\mathrm{V}2} + x''^2x^{\mathrm{IV}3} + x'''^2x^{\mathrm{V}3})$$

$$+20\left(\begin{array}{c} x'^3x'''x^{\mathrm{V}} + x'x''^3x^{\mathrm{IV}} + x'x'''x^{\mathrm{IV}3} + x''x'''^3x^{\mathrm{V}} \\ + x''x^{\mathrm{IV}}x^{\mathrm{V}3} \end{array}\right)$$

$$+30\left(\begin{array}{c} x'^2x''^2x^{\mathrm{V}} + x'x''^2x'''^2 + x'^2x^{\mathrm{IV}}x^{\mathrm{V}2} + x''x'''^2x^{\mathrm{IV}2} \\ + x'''x^{\mathrm{IV}2}x^{\mathrm{V}2} \end{array}\right)$$

$$+60\left(\begin{array}{c} x'^2x''x'''x^{\mathrm{IV}} + x'x''x'''x^{\mathrm{V}2} + x'x''x^{\mathrm{IV}2}x^{\mathrm{V}} \\ + x'x'''^2x^{\mathrm{IV}}x^{\mathrm{V}} + x'^2x'''x^{\mathrm{IV}}x^{\mathrm{V}} \end{array}\right)$$

$$\xi''' =$$

$$5\,(x'^4x''' + x''^4x^{\mathrm{IV}} + x'x^{\mathrm{IV}4} + x'''^4x^{\mathrm{V}} + x''x^{\mathrm{V}4})$$

$$+10\,(x'^3x''^2 + x''^3x'''^2 + x'^2x^{\mathrm{V}3} + x'''^3x^{\mathrm{IV}2} + x^{\mathrm{IV}3}x^{\mathrm{V}2})$$

$$+20\left(\begin{array}{c} x'^3x^{\mathrm{IV}}x^{\mathrm{V}} + x'x''^3x^{\mathrm{V}} + x'x''x'''^3 + x''x'''x^{\mathrm{IV}3} \\ + x'''x^{\mathrm{IV}}x^{\mathrm{V}3} \end{array}\right)$$

$$+30\left(\begin{array}{c} x'^2x''x^{\mathrm{IV}2} + x'^2x'''^2x^{\mathrm{IV}} + x'x'''^2x^{\mathrm{V}2} + x''^2x'''x^{\mathrm{V}2} \\ + x''^2x^{\mathrm{IV}2}x^{\mathrm{V}} \end{array}\right)$$

$$+60\left(\begin{array}{c} x'^2x''x'''x^{\mathrm{V}} + x'x''^2x'''x^{\mathrm{IV}} + x'x''x^{\mathrm{IV}}x^{\mathrm{V}2} \\ + x'x'''x^{\mathrm{IV}2}x^{\mathrm{V}} + x''x'''^2x^{\mathrm{IV}}x^{\mathrm{V}} \end{array}\right)$$

$$\xi^{\prime v} =$$

$$5\,(x^{\prime 4}x^{\prime v} + x^{\prime}x^{\prime\prime\prime 4} + x^{\prime\prime 4}x^{v} + x^{\prime\prime}x^{\prime v 4} + x^{\prime\prime\prime}x^{v 4})$$
$$+10\,(x^{\prime 2}x^{\prime\prime 3} + x^{\prime 3}x^{v 2} + x^{\prime\prime 2}x^{\prime\prime\prime 3} + x^{\prime\prime\prime 2}x^{\prime v 3} + x^{\prime v 2}x^{v 3})$$
$$+20\begin{pmatrix} x^{\prime 3}x^{\prime\prime}x^{\prime\prime\prime} + x^{\prime\prime 3}x^{\prime\prime\prime}x^{\prime v} + x^{\prime}x^{\prime\prime}x^{v 3} + x^{\prime}x^{\prime v 3}x^{v} \\ + x^{\prime\prime\prime 3}x^{\prime v}x^{v} \end{pmatrix}$$
$$+30\,(x^{\prime 2}x^{\prime\prime\prime 2}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime v 2} + x^{\prime\prime 2}x^{\prime v}x^{v 2})$$
$$+60\begin{pmatrix} x^{\prime 2}x^{\prime\prime}x^{\prime v}x^{v} + x^{\prime 2}x^{\prime\prime\prime}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime}x^{v 2} \\ + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime v} + x^{\prime}x^{\prime\prime\prime}x^{\prime v}x^{v 2} \end{pmatrix}$$

$$\xi^{v} =$$

$$5\,(x^{\prime 4}x^{v} + x^{\prime}x^{\prime\prime 4} + x^{\prime\prime}x^{\prime\prime\prime 4} + x^{\prime\prime\prime}x^{\prime v 4} + x^{\prime v}x^{v 4})$$
$$+10\,(x^{\prime 3}x^{\prime\prime 2} + x^{\prime 2} + x^{\prime v 3} + x^{\prime\prime 3}x^{\prime v 2} + x^{\prime\prime 2}x^{v 3} + x^{\prime\prime\prime 3}x^{v 2})$$
$$+20\begin{pmatrix} x^{\prime 3}x^{\prime\prime}x^{\prime v} + x^{\prime}x^{\prime\prime\prime 3}x^{\prime v} + x^{\prime\prime 3}x^{\prime\prime\prime}x^{v} + x^{\prime}x^{\prime\prime\prime}x^{v 3} \\ + x^{\prime\prime}x^{\prime v 3}x^{v} \end{pmatrix}$$
$$+30\begin{pmatrix} x^{\prime 2}x^{\prime\prime}x^{v\,2} + x^{\prime}x^{\prime\prime 2}x^{v 2} + x^{\prime\prime 2}x^{\prime\prime\prime 2}x^{\prime v} \\ + x^{\prime}x^{\prime v 2}x^{v 2} + x^{\prime\prime\prime 2}x^{\prime v 2}x^{v} \end{pmatrix}$$
$$+60\begin{pmatrix} x^{\prime 2}x^{\prime\prime\prime}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime 2}x^{v} \\ + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime}x^{\prime v 2} + x^{\prime\prime}x^{\prime\prime\prime}x^{\prime v}x^{v 2} \end{pmatrix}$$

3. Now, if in the expression $\xi^{\prime} + \xi^{\prime\prime}\alpha + \xi^{\prime\prime\prime}\alpha^2 + \xi^{\prime v}\alpha^3 + \xi^{v}\alpha^4$, we perform all the possible transpositions of the roots $x^{\prime}$, $x^{\prime\prime}$, $x^{\prime\prime\prime}$, $x^{\prime v}$, and denote the sum of all the results thus obtained by

$$\zeta^{\prime} + \zeta^{\prime\prime}\alpha + \zeta^{\prime\prime\prime}\alpha^2 + \zeta^{\prime v}\alpha^3 + \zeta^{v}\alpha^4$$

we then find

$$\zeta^{\prime} = 24\,[5] + 8\,.\,20\,[1^2 3] + 8\,.\,30\,[12^2] + 24\,.\,120\,[1^5]$$
$$\zeta^{\prime\prime} = 6\,.\,5\,[14] + 6\,.\,10\,[23] + 4\,.\,20\,[1^2 3] + 4\,.\,30\,[12^2] + 6\,.\,60\,[1^3 2]$$

and for $\zeta^{\prime\prime\prime}$, $\zeta^{\prime v}$, ζ^{v}, the same values as for $\zeta^{\prime\prime}$. Therefore

$$\zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta^{iv}\alpha^3 + \zeta^{v}\alpha^4 =$$
$$\zeta' + \zeta''(\alpha + \alpha^2 + \alpha^3 + \alpha^4) = \zeta' - \zeta'';$$

because $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = [1] - 1 = -1.$

4. But by 13 of the preceding § the coefficient p'' is the sum of all the values of forms of $(f: (12345))^5$, which arise from the transposition of the four first roots; we have ∴ also

$$p'' = \zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta^{iv}\alpha^3 + \zeta^{v}\alpha^4 = \zeta' - \zeta''.$$

Now, if we substitute for ζ', ζ'', their values already found, we then get

$$p'' = 24\,[5] - 30\,[14] - 60\,[23] + 80\,[1^23] + 120\,[12^2] - 360\,[1^32] + 2880\,[1^5]$$

and when for the numerical expressions we substitute their values taken from the annexed Tables,

$$p'' = 24\,A^5 - 150\,A^3B + 150\,AB^2 + 250\,A^2C - 250\,BC - 1250\,AD + 6250\,E.$$

5. By a method not much different from this, we may also find the coefficient q''. Having, however, found p'' and q'', then the solution of the equation IV in 10 of the foregoing §, gives the value of p'. Having obtained p', then we may also find the coefficients q', r' of the equation III; and the solution of this equation gives the value of p. From the known value of p we may now again find the coefficients q, r, s, of the equation I. But the calculations by this method would be extremely troublesome, and almost impracticable. I shall, in the third part of this collection, show how it may be shortened essentially, and at the same time give the complete

solution of equations of the fifth, sixth, and seventh degrees.

6. Let K', K'', K''', K'^{v}, be the four roots of the equation II, consequently

$$f': (12345) = K', f': (23415) = K''$$
$$f': (34125) = K''', f': (41235) = K'^{v}.$$

If we substitute K', K'', K''', K'^{v}, for K in the equation I, we then get for t four values $\sqrt[5]{K'}$, $\sqrt[6]{K''}$, $\sqrt[5]{K'''}$, $\sqrt[5]{K'^{v}}$, and the values of forms f: (12345), f: (23415), f: (34125), f: (41235) correspond to these values; we ∴ have

$$f: (12345) = \sqrt[5]{K'}, f: (23415) = \sqrt[5]{K''}$$
$$f: (34125) = \sqrt[5]{K'''}, f: (41235) = \sqrt[5]{K'^{v}}.$$

If we substitute here for f: (12345), its value $x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v} + \alpha^4 x^{v}$, we then get, including the equation $x' + x'' + x''' + x'^{v} + x^{v} = A$, the five following equations:

$$x' + x'' + x''' + x'^{v} + x^{v} = A$$
$$x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v} + \alpha^4 x^{v} = \sqrt[5]{K'}$$
$$x'' + \alpha x''' + \alpha^2 x'^{v} + \alpha^3 x' + \alpha^4 x^{v} = \sqrt[5]{K''}$$
$$x''' + \alpha x'^{v} + \alpha^2 x' + \alpha^3 x'' + \alpha^4 x^{v} = \sqrt[5]{K'''}$$
$$x'^{v} + \alpha x' + \alpha^2 x'' + \alpha^5 x''' + \alpha^4 x^{v} = \sqrt[5]{K'^{v}}.$$

7. If we multiply the four last equations by α, and then add them to the first, since $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = `[1] = 0$, we get immediately

$$x = \frac{A + \alpha(\sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{IV}})}{5}$$

and the remaining roots x', x'', x''', x^{IV}, are all of the form $aA + b\sqrt[5]{K'} + c\sqrt[5]{K''} + d\sqrt[5]{K'''} + e\sqrt[5]{K^{IV}}$ in which a, b, c, d, e, denote certain functions of α. Hence now we may conclude, in a similar way as in equations of the third and fourth degrees, that

$$x = \frac{A + \sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{IV}}}{5}$$

is a root of the given equation.

8. Having ∴ solved the equation II, we immediately have a root of the given equation, and the remaining roots may be determined from the five equations in 6, by elimination, if, after having performed the calculation, we merely substitute $\alpha^4\sqrt[5]{K'}$, $\alpha^4\sqrt[5]{K''}$, $\alpha^4\sqrt[5]{K'''}$, $\alpha^4\sqrt[5]{K^{IV}}$ for $\sqrt[5]{K'}$, $\sqrt[5]{K''}$, $\sqrt[5]{K'''}$, $\sqrt[5]{K^{IV}}$.

SECTION CLXIV.

Prob. Find functions, which are fit for the solution of the general equation of the sixth degree

$$x^6 - Ax^5 + Bx^4 - Cx^3 + Dx^2 - Ex + F = 0$$

under the supposition, that we know not how to solve any other equations than those of lower degrees, and those of the form $t^6 - K = 0$.

Solution 1. With the view of arranging the $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$ values of forms of f: (123456) in a re-

curring order, if we put the 120 values of forms in 1, § CLXII, in a vertical column under each other, and add to each the root x^{VI} in the last place, we then have 120 values of forms, all of which end with x^{VI}. From each of these, if we derive, by a recurring transposition of all the six roots, five others, we then get 120 periods, each consisting of six values of forms, consequently the 720 values of forms of $f: (123456)$.

2. I shall now assume, that the six values of forms of the first period $f: (123456)$, $f: (234561)$, $f: (345612)$, $f: (456123)$, $f: (561234)$, $f: (612345)$, are the roots of the equation

$$t^6 - K = 0;$$

then $-K$ is the product of all these roots, and hence

$$-K = f: (123456) \times f: (234561) \times f: (345612) \times f: (456123) \times f: (561234) \times f: (612345).$$

This product, however, evidently undergoes no change in each recurring transposition of all the six roots x', x'', x''', x^{IV}, x^{V}, x^{VI}, because in each such transposition, one factor merely changes place with another; consequently the 720 values of forms of K, taken six and six together, are equal. Therefore K can have no more than 120 different values, and these 120 values are no other than those which have x^{VI} in the last place, and which consequently arise merely from the transposition of the five remaining roots.

3. Since $K = t^6$, and $t = f: (123456)$, then also $K = (f: (123456))^6$. Consequently the function $(f: (123456))^6$ can have no more different values than

those which exclusively arise from the transposition of the roots x', x'', x''', x^{IV}, x^{V}. For shortness' sake, I shall denote them by f': (123456), and assume, that the five values of forms f': (123456), f': (234516), f': (345126), f': (451236), f': (512346), which arise from the recurring transpositions of the five first roots x', x'', x''', x^{IV}, x^{V}, are the roots of the following equation of the fifth degree:

$$K^5 - pK^4 + qK^3 - rK^2 + sK - u = 0,$$

then p, q, r, s, u, are symmetrical functions of the above values of forms, and consequently undergo no change in each recurring transposition of the first five roots. But since they also remain the same in the recurring transpositions of all the six roots, they consequently can contain no more different values, than those which arise exclusively from the transpositions of the roots x', x'', x''', x^{IV}. Therefore each of these functions depends only on an equation of the 24th degree. Since they are homogeneous, it will be sufficient to have determined one of these functions.

4. If for the sake of brevity, we put $p = f''$: (123456), then f'': (123456) can only undergo a change when the four first roots are transposed. I shall now assume, that the four values of forms f'': (123456), f'': (234156), f'': (341256), f'': (412356), are the roots of the following equation of the fourth degree:

$$p^4 - p'p^3 + q'p^2 - r'p + s' = 0;$$

then the coefficients p', q', r', s', are symmetrical functions of these values of forms, and consequently in each

recurring transposition of the four first roots they remain unchanged; and since they also remain the same in the recurring transpositions of the first five and of all the six roots, consequently, amongst the 720 values of forms, there are no more than six which are different, viz. those which arise exclusively from the transposition of the three first roots x', x'', x'''.

5. I put $p' = f''' : (123456)$, and assume that the three values of forms $f''' : (123456)$, $f''' : (231456)$, $f : (312456)$, are the roots of the following equation of the third degree:

$$p'^3 - p''p'^2 + q''p' - r'' = 0;$$

then p'', q'', r'', are symmetrical functions of these values of forms, and consequently in the recurring transpositions of the three first roots suffer no change; and since they also remain unchanged in the recurring transpositions of the four and five first, likewise of all the six roots, ∴ each of these functions can have no more than two different values, viz. those which arise from the transposition of the two first roots.

6. If ∴ we put $p'' = f^{IV} : (123456)$, and assume that the two values of forms $f^{IV} : (123456)$, $f^{IV} : (213456)$, are the roots of the equation

$$p''^2 - p'''p'' + q''' = 0,$$

then the functions p''', q''' undergo no change in the transposition of the two first, three first, four first, five first, and all the six roots, consequently they are symmetrical, and ∴ may be expressed rationally by means of the coefficients A, B, C, D, E, F, of the given equation.

7. The equation for t, which originally was of the 720th degree, has consequently been reduced by these successive operations to the following equations:

$$\begin{aligned}
&\text{I.}\ \ t^6 - K = 0\\
&\text{II.}\ \ K^5 - pK^4 + qK^3 - rK^2 + sK - u = 0\\
&\text{III.}\ p^4 - p'p^3 + q'p^2 - r'p + s' = 0\\
&\text{IV.}\ \ p'^3 - p''p'^2 + q''p' - r'' = 0\\
&\text{V.}\ \ p''^2 - p'''p'' + q''' = 0
\end{aligned}$$

which are so constituted, that the coefficients of each of them depend on the solution of all the following equations. The equation V gives two values of p'', ∴ the equation IV six values of p', consequently the equation III 24 values of p, and ∴ the equation II 120 values of K, consequently the equation I 720 values of t.

8. Therefore, if the function t be such, that the values of forms $f: (123456)$, $f: (234561)$, $f: (345612)$, $f: (456123)$, $f: (561234)$, $f: (612345)$, are the roots of the equation $t^6 - K = 0$, consequently it is always fit for the solution of an equation of the 6th degree. But nothing more will be required to effect this, than that

$$f: (123456) = \alpha f: (234561) = \alpha^2 f: (345612)$$
$$= \alpha^3 f: (456123) = \alpha^4 f: (561234) = \alpha^5 f: (612345)$$

when α denotes a primitive root of the equation $t^6 - 1 = 0$. But all functions of the form

$$\phi: (123456) + \alpha\phi: (234561) + \alpha^2\phi: (345612)$$
$$+ \alpha^3\phi: (456123) + \alpha^4\phi: (561234) + \alpha^5\phi: (612345)$$

are of this nature. Consequently all functions of this

kind are fit for the solution of equations of the 6th degree. Besides, we need not be apprehensive of not being able to determine the roots of the given equation from these functions; for, since the equal values of forms of $f:(123456)$, if indeed it should have any, can only be found amongst those which have x^{VI} in the last place, consequently x^{VI} cannot be amongst the roots, which correspond to the equal values of $t = f:(123456)$, and $\therefore$, by the foregoing chapter, this root at least must be determined from the known value of t by a rational expression.

SECTION CLXV.

PROB. To solve actually the general equation of the 6th degree in the foregoing §.

Solution 1. If, to facilitate the operation, we put $\phi:(123456) = x'$, then
$f:(123456) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V} + \alpha^5 x^{VI}$.
We have $\therefore$

$$f':(123456) = \big(f:(123456)\big)^6$$
$$= (x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V} + \alpha^5 x^{VI})^6 =$$
$$\alpha^{-6}(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + \alpha^5 x^{V} + \alpha^6 x^{VI})^6$$
$$= (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + \alpha^5 x^{V} + \alpha^6 x^{VI})^6.$$

This transformation was performed merely in order to make the dashes over x agree with the powers of α, by which means the solution of the polynomial is rendered more easy.

2. By 2 of the foregoing §, $f'':(123456)$ is the sum

of all the results which are obtained from the recurring transpositions of the five first roots in f^{v} : (123456), consequently also the sum of all the results, which arise from the transposition of the five first roots in $(f: (123456))^6$. Further, by 3 of the foregoing §, $f^{\prime\prime\prime}$: (123456) is the sum of all the recurring transpositions of the four first roots in $f^{\prime\prime}$: (123456), consequently also the sum of all the recurring transpositions of the four and five first roots in $(f: (123456))^6$. By 4, $f^{\prime v}$: (123456) is the sum of all the recurring transpositions of the three first roots in $f^{\prime\prime\prime}$: (123456), consequently also the sum of all the recurring transpositions of the three first, four first, and five first roots in $(f: (123456))^6$. Now, since by 5, $p^{\prime\prime\prime} = f^{\prime v}$: (123456) $+ f^{\prime v}$: (213456), ∴ also $p^{\prime\prime\prime}$ is the sum of all the recurring transpositions of the two first, three first, four first, and five first roots in $(f: (123456))^6$, consequently the sum of all the values, which are obtained from this function by the transposition of the five first roots.

3. In order ∴ to find the coefficient $p^{\prime\prime\prime}$, we must first solve the power

$$(\alpha x^{\prime} + \alpha^2 x^{\prime\prime} + \alpha^3 x^{\prime\prime\prime} + \alpha^4 x^{\prime v} + \alpha^5 x^{v} + \alpha^6 x^{v\prime})^6;$$

this solution, since $\alpha^6 = 1$, $\alpha^7 = \alpha$, $\alpha^8 = \alpha^2$, &c. assumes the following form

$$\zeta^{\prime} + \zeta^{\prime\prime}\alpha + \zeta^{\prime\prime\prime}\alpha^2 + \zeta^{\prime v}\alpha^3 + \zeta^{v}\alpha^4 + \zeta^{v\prime}\alpha^5$$

in which $\zeta^{\prime}$, $\zeta^{\prime\prime}$, $\zeta^{\prime\prime\prime}$, $\zeta^{\prime v}$, ζ^{v}, $\zeta^{v\prime}$, are functions of $x^{\prime}$, $x^{\prime\prime}$, $x^{\prime\prime\prime}$, $x^{\prime v}$, x^{v}, $x^{v\prime}$, without α. If we then perform the 120 transpositions of the roots $x^{\prime}$, $x^{\prime\prime}$, $x^{\prime\prime\prime}$, $x^{\prime v}$, x^{v}, then the sum of all the results thus obtained, gives the coefficient $p^{\prime\prime\prime}$.

4. Further, if we multiply the two functions $f''':$ (123456), $f''':$ (213456) together, we then obtain also the coefficient q'''. The equation V of the foregoing §, will then give the coefficient p''; and having found this, we may directly find q'', r'' by the foregoing chapter. Now, the equation IV gives the coefficient p', ∴ also q', r', s', and lastly the equation III gives the coefficient p, and at the same time also the remaining coefficients of the equation II.

5. Let K', K'', K''', K^{IV}, K^{V}, denote the five roots of the equation II; then $\sqrt[6]{K'}$, $\sqrt[6]{K''}$, $\sqrt[6]{K'''}$, $\sqrt[6]{K^{IV}}$, $\sqrt[6]{K^{V}}$, are the five values of t, and the values of forms $f:$ (123456), $f:$ (234516), $f:$ (345126), $f:$ (451236), $f:$ (512346) correspond to these values; we ∴ have the six equations

$$x' + x'' + x''' + x^{IV} + x^{V} + x^{VI} = A$$
$$x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V} + \alpha^5 x^{VI} = \sqrt[6]{K'}$$
$$x'' + \alpha x''' + \alpha^2 x^{IV} + \alpha^3 x^{V} + \alpha^4 x' + \alpha^5 x^{VI} = \sqrt[6]{K''}$$
$$x''' + \alpha x^{IV} + \alpha^2 x^{V} + \alpha^3 x' + \alpha^4 x'' + \alpha^5 x^{VI} = \sqrt[6]{K'''}$$
$$x^{IV} + \alpha x^{V} + \alpha^2 x' + \alpha^3 x'' + \alpha^4 x''' + \alpha^5 x^{VI} = \sqrt[6]{K^{IV}}$$
$$x^{V} + \alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + \alpha^5 x^{VI} = \sqrt[6]{K^{V}}.$$

6. Hence now, when we multiply the five last equations by α, and add them to the first, we immediately obtain

$$x^{VI} = \frac{A + \alpha(\sqrt[6]{K'} + \sqrt[6]{K''} + \sqrt[6]{K'''} + \sqrt[6]{K^{IV}} + \sqrt[6]{K^{V}})}{6}$$

and when for the same reasons as in the case of lower equations, we put $a^5\sqrt[6]{K'}$, $a^5\sqrt[6]{K''}$, $a^5\sqrt[6]{K'''}$, $a^5\sqrt[6]{K^{iv}}$, $a^5\sqrt[6]{K^{v}}$, for $\sqrt[6]{K'}$, $\sqrt[6]{K''}$, $\sqrt[6]{K'''}$, $\sqrt[6]{K^{iv}}$, $\sqrt[6]{K^{v}}$, so that a vanish, we get a root of the given equation

$$x = \frac{A+\sqrt[6]{K'}+\sqrt[6]{K''}+\sqrt[6]{K'''}+\sqrt[6]{K^{iv}}+\sqrt[6]{K}}{6}$$

and we obtain the remaining ones from the above six equations by elimination.

SECTION CLXVI.

Prob. Solve the general equation of the nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0.$$

Solution 1. Let $t = f: (12345 \ldots\ldots n)$ be a function of such a nature, that the n values of forms, which arise from the recurring transposition of all the n roots, are the roots of the equation

$$t^n - K = 0.$$

Since then K is the product of all these values of forms, it must consequently remain the same in all these transpositions. Therefore its values of forms, $1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n$, taken n and n together, are equal. Hence it follows, that K has no more than $1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n-1$ different values, and that these values are those which arise exclusively from the transposition of the $n - 1$ first roots.

2. Since $\therefore$ K still depends on an equation of the $1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n - 1$th degree, we must consequently

endeavour to reduce this equation. To effect this, I put $K = f' : (12345 \ldots\ldots n)$, and assume that the $n - 1$ values of forms of this function, which arise from a recurring transposition of the $n - 1$ first roots, are the roots of the following equation:

$$K^{n-1} = pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0.$$

Then the coefficients p, q, r, &c. are symmetrical functions of these values of forms, and consequently in the recurring transpositions of the $n - 1$ first roots remain the same. But since they also undergo no change in the recurring transpositions of all the n roots, because by that means $f : (1234 \ldots\ldots n)$ suffers no change, consequently only those of their values are different, which arise from the transposition of the $n - 2$ first roots. Therefore each of these coefficients has only $1 . 2 . 3 . 4 \ldots\ldots n - 2$ different values, and consequently each of them depends on an equation of the $1 . 2 . 3 . 4 \ldots\ldots n - 2$th degree only.

3. Since it is quite sufficient to have found p, because p, q, r, &c. are homogeneous functions, I shall put $p = f'' : (12345 \ldots\ldots n)$, and assume, that the $n - 2$ values of forms of this function, which arise from a recurring transposition of the $n - 2$ first roots, are the roots of the equation

$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} + \&c. = 0;$$

then p', q', r', &c. are symmetrical functions of these values, and they ∴ remain the same in the recurring transpositions of the $n - 2$, $n - 1$ first roots, and also of all the n roots. Therefore these coefficients have only $1 . 2 . 3 . 4 \ldots\ldots n - 3$ different values, viz. those which

arise from the transposition of the $n-3$ first roots, and they consequently depend only on equations of the $1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots n-3$th degree.

4. In a similar way we successively form the equations

$$\begin{array}{l}
p'^{n-3} - p'' p'^{n-4} + q'' p'^{n-5} - r'' p'^{n-6} + \&c. = 0 \\
p''^{n-4} - p''' p''^{n-5} + q''' p''^{n-6} - r''' p''^{n-7} + \&c. = 0 \\
p'''^{n-5} - p'^{V} p'''^{n-6} + q'^{V} p'''^{n-7} - r'^{V} p'''^{n-8} + \&c. = 0 \\
\qquad\qquad\qquad \&c.
\end{array}$$

viz. the first from the recurring period of the $n-3$ first roots of the function $p' = f''' : (1234 \ldots\ldots. n)$; the second from the recurring period of the $n-4$ first roots of the function $p'' = f'^{V} : (1234 \ldots\ldots\ldots n)$; the third from the recurring period of the $n-5$ first roots of the function $p''' = f^{V} : (1234 \ldots\ldots n)$; and so on. We continue this operation till we arrive at an equation of the second degree

$$(p^{(n-4)})^2 - p^{(n-3)} p^{(n-4)} + q^{(n-3)} = 0;$$

then $p^{(n-3)}$, $q^{(n-)}$, are such functions of x', x'', x''', ... $x^{(n)}$, as remain the same in the recurring transpositions of all the n roots, also in the recurring transpositions of the $n-1$, $n-2$, $n-3$ first roots, and so on, and likewise of the two first roots. They $\therefore$ undergo no change in all the transpositions of the roots, and consequently they are symmetrical. Therefore they may be expressed rationally by the coefficients of the given equation.

5. We consequently have a series of equations

$$t^n - K = 0$$

$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$

$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} + \&c. = 0$$

$$p'^{n-3} - p''p'^{n-4} + q''p'^{n-5} - r''p'^{n-6} + \&c. = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(p^{(n-4)})^2 - p^{(n-3)}p^{(n-4)} + q^{(n-3)} = 0$$

which are so constituted, that the first coefficient of each depends on all the following ones. Now if the first coefficients p, p', p'', p''', &c. are found, then also the remaining ones q, r, &c. q', r', &c. q'', r'', &c. &c., may be found by the foregoing chapter.

6. It only remains now to assume for $t = f: (12345\ldots\ldots n)$ a function, such, that its values of forms, which arise from the recurring transposition of all the n roots x', x'', x''', &c. may be the roots of an equation of the form $t^n - K = 0$. With this view, we assume any other function $z = \phi: (12345 \ldots\ldots x)$ at pleasure. Let $z', z'', z''', z^{IV}, \ldots\ldots z^{(n)}$ denote the values of forms of z, which arise from the recurring transposition of all the n roots, and α a primitive root of the equation $x^n - 1 = 0$: assert, that then

$$t = z' + \alpha z'' + \alpha^2 z''' + \alpha^3 z^{IV} + \ldots\ldots + \alpha^{n-1} z^{(n)}$$

is always a function of the required property. For since in this function, in each recurring transposition of the roots x', x'', x''', &c., the values of forms z', z'', z''', $z^{IV}, \ldots\ldots z^{(n)}$ are in like manner transposed in a recurring order (for by these means z' changes place with z'', z'' with z''', z''' with z^{IV}, and so on, lastly $z^{(n)}$ with z'), there-

fore the function t, in the recurring transpositions of the roots x', x'', x''', &c. has the following values:

$$t' = z' + \alpha z'' + \alpha^2 z''' + \alpha^3 z^{IV} + \dots\dots\dots + \alpha^{(n-1)} z^{(n)}$$
$$t'' = z'' + \alpha z''' + \alpha^2 z^{IV} + \alpha^3 z^{V} + \dots\dots\dots + \alpha^{(n-1)} z'$$
$$t''' = z''' + \alpha z^{IV} + \alpha^2 z^{V} + \alpha^3 z^{VI} + \dots\dots\dots + \alpha^{(n-1)} z'$$
$$t^{IV} = z^{IV} + \alpha z^{V} + \alpha^2 z^{VI} + \alpha^3 z^{VII} + \dots\dots\dots + \alpha^{(n-1)} z'''$$
&c.

and we immediately see, that $t'' = \alpha^{n-1} t'$, $t''' = \alpha^{n-2} t'$, $t^{IV} = \alpha^{n-3} t'$, and so on. Therefore the functions t', t'', t''', &c. have exactly those relations which they ought to have, in order that they may be the roots of an equation of the form $t^n - K = 0$.

7. That the value of the roots x', x'', x''', &c., may always be determined from the known value of the function $z' + \alpha z'' + \alpha^2 z''' + \alpha^3 z^{IV} + \dots\dots\dots + \alpha^{n-1} z^{(n)}$ let the function z be what it may, appears from this, that the root $x^{(n)}$ never can correspond to the equal values of this function, if it should have any, because these equal values must necessarily be amongst those which arise from the transposition of the $n-1$ roots x', x'', x''', x^{IV}, $\dots\dots\dots x^{(n-1)}$. Consequently this one root at least may always be determined from the known value of t, without solving any equation. But then the remaining ones may also be found, when the solution of equations below the nth degree is pre-supposed.

8. Therefore all equations may be found in numberless ways. If, in order to make the calculation more simple, we put $z = x$, we then have

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \dots\dots\dots + \alpha^{n-1} x^{(n)}.$$

Hence we immediately obtain

$$K = t^n = (x' + \alpha x'' + \alpha^2 x''' + \ldots\ldots + \alpha^{n-1}x^{(n)})^n$$

9. Now, in order to solve actually the given equation, we must first of all endeavour to determine the coefficients $p^{(n-3)}$, $q^{(n-3)}$ of the last reduced equation in 4. Since p is the sum of the $n-1$ values of forms of K, which arise from the recurring transposition of the $n-1$ first roots, and p' again the sum of all the values of forms of p, which arise from the recurring transposition of the $n-2$ first roots; consequently, also, p' is the sum of the $n-1.n-2$ values of forms of K, which arise from the recurring transpositions of the $n-1$ and of the $n-2$ first roots. Further, since p'' is the sum of the $n-3$ values of forms of p', which arise from the recurring transposition of the $n-3$ first roots, ∴ also p'' is the sum of the $n-1.n-2.n-3$ values of forms of K, which arise from the recurring transpositions of the $n-1$, $n-2$, and $n-3$ first roots. If we proceed further in this way, we then find that the coefficient $p^{(n-3)}$ is the sum of all the $n-1.n-2.n-3....3.2$ values of forms of K, which arise from the recurring transpositions of the $n-1$, $n-2$, $n-3$ roots, and so on, lastly, of the two first roots, or, which is the same, that $p^{(n-3)}$ is the sum of all the $1.2.3.4...n-1$ values of forms of K, which arise from all the transpositions of the $n-1$ first roots.

10. In order, therefore, to find the coefficient $p^{(n-3)}$, we must in the first place solve the expression for K in 8

$$(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \ldots\ldots + \alpha^{n-2}x^{(n)})$$

according to the powers of α. This solution, since $\alpha^n = 1$, $\alpha^{n+1} = \alpha$, $\alpha^{n+2} = \alpha^2$, &c. will then have the following form:

$$\xi + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \ldots\ldots + \xi^{(n)}\alpha^{n-1};$$

in which ξ', ξ'', ξ''',......$\xi^{(n)}$ are certain functions of x', x'', x''',......$x^{(n)}$. Now, if in these we transpose the roots x', x'', x''', ... $x^{(n-1)}$ in all possible ways, while $x^{(n)}$ retains its place, and then add the results together, we obtain an expression of the form

$$\zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta'\ \alpha^3 + \ldots + \zeta^{(n)}\alpha^{(n-1)}$$

which, since it is the value of $p^{(n-3)}$, is necessarily symmetrical with reference to the roots x', x'', x'''......$x^{(n)}$, and $\therefore$ may be expressed rationally by the coefficients A, B, C &c. of the given equation.

11. In order to find the coefficient $q^{(n-3)}$ of the last equation in 5, in the expression $\xi' + \xi''\alpha + \xi'''\alpha^2 + \ldots + \xi^{(n)}\alpha^{(n-1)}$ complete the $n.n-1.n-2......3$ recurring transpositions of the $n-1$, $n-2$, $n-3$, and so on, first roots, exclusive of the two first; then substitute x' for x'', and again make the same transpositions. Now, if we multiply the sum of the $n.n-1.n-2........3$ first results by the sum of the $n.n-1.n-2......3$ last, we then obtain a symmetrical function of x', x'', x''',...$x^{(n)}$, which will give the value of $q^{(n-3)}$, expressed by the coefficients A, B, C, &c.

12. If we write the reduced equations in 5 backwards,

we have the following series:

$$(p^{(n-4)})^2 - p^{(n-3)}(p^{(n-4)}) \;+ q^{(n-3)} = 0$$
$$(p^{(n-5)})^3 - p^{(n-4)}(p^{(n-5)})^2 + q^{(n-4)}(p^{(n-5)}) - r^{(n-4)} = 0$$
$$(p^{(n-6)})^4 - p^{(n-5)}(p^{(n-6)})^3 + q^{(n-5)}(p^{(n-6)})^2 - r^{(n-5)}(p^{(n-6)}) + s^{(n-5)} = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$
$$t^n - K = 0.$$

Having found the coefficients $p^{(n-3)}$, $q^{(n-3)}$ by 10 and 11, then the solution of the first equation gives the coefficient $p^{(n-4)}$ of the second equation; and by the foregoing chapter, we may hence immediately determine the values of $q^{(n-4)}$ and $r^{(n-4)}$ by rational expressions, because $p^{(n-4)}$, $q^{(n-4)}$ and $r^{(n-4)}$, are homogeneous functions. The coefficients of the second equation are consequently fully determined, and its solution gives the coefficient $p^{(n-5)}$ of the third equation; $\therefore$ also, as before, the coefficients $q^{(n-5)}$, $r^{(n-5)}$, $s^{(n-5)}$; and the solution of this last equation again gives the coefficient $p^{(n-6)}$ of the following equation. If we proceed further in this way, we at last find the coefficients p, q, r, &c. of the equation for K.

13. Let K', K'', K''', K^{IV}......$K^{(n-1)}$, be the $n-1$ roots of this equation; corresponding to these, as we have assumed in 2, are those values of forms of $K = f^7 : (1234...n)$, which arise from the recurring transposition of the $n-1$ first roots. Each of these values of K substituted in the equation $t^n - K = 0$, gives n values of t, consequently all together they give $n.n-1$ values of t; and these are:

$$\sqrt[n]{K'},\ \alpha\sqrt[n]{K'},\ \alpha^2\sqrt[n]{K'},\dots\dots\dots\alpha^{n-1}\sqrt[n]{K'}$$
$$\sqrt[n]{K''},\ \alpha\sqrt[n]{K''},\ \alpha^2\sqrt[n]{K''},\dots\dots\dots\alpha^{n-1}\sqrt[n]{K''}$$
$$\sqrt[n]{K'''},\ \alpha\sqrt[n]{K'''},\ \alpha^2\sqrt[n]{K'''},\dots\dots\dots\alpha^{n-1}\sqrt[n]{K'''}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\sqrt[n]{K^{(n-1)}},\ \alpha\sqrt[n]{K^{(n-1)}},\ \alpha^2\sqrt[n]{K^{(n-1)}}\dots\alpha^{n-1}\sqrt[n]{K^{(n-1)}}$$

and corresponding to these are the $n \,.\, n-1$ values of forms of $t = f\colon (1234\dots\dots n)$, which arise from the recurring transposition of all the n and $n-1$ first roots. These numerical values of the function t have such a relation to the values of forms of $f\colon (1234\dots\dots n)$, that if we put $f\colon (1234\dots\dots n) = \alpha^{\nu}\sqrt[n]{K'}$, the remaining $n-2$ values of forms, which arise from $f\colon (1234\dots\dots n)$ by the recurring transpositions of the $n-1$ first roots, correspond to the numerical values $\alpha^{\nu}\sqrt[n]{K''}$, $\alpha^{\nu}\sqrt[n]{K'''}$, $\alpha^{\nu}\sqrt[n]{K^{\prime v}},\dots\dots\alpha^{\nu}\sqrt[n]{K^{(n-1)}}$. Now since we have assumed that $t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \dots\dots + \alpha^{n-1}x^{(n)}$, we have $\therefore$ the following n equations:

$$\begin{aligned}
&x' + x'' + x''' + \dots + x^{(n-1)} + x^{(n)} = A\\
&x' + \alpha x'' + \alpha^2 x''' + \dots + \alpha^{n-2}x^{(n-1)} + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K'}\\
&x'' + \alpha x''' + \alpha^2 x^{\prime v} + \dots + \alpha^{n-2}x' \quad + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K''}\\
&x''' + \alpha x^{\prime v} + \alpha^2 x \quad + \dots + \alpha^{n-2}x'' \quad + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K'''}\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&x^{(n-1)} + \alpha x' + \alpha^2 x'' + \dots + \alpha^{n-2}x^{(n-2)} + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K^{(n-1)}}.
\end{aligned}$$

14. If we multiply the $n-1$ last equations by α, and

then add them to the first, we obtain, after dividing by n,

$$x^{(n)} = \frac{A}{n} + \frac{\alpha^{\nu+1}}{n}(\sqrt[n]{K'} + \sqrt[n]{K''} + \sqrt[n]{K'''} + \ldots + \sqrt[n]{K^{(n-1)}})$$

and the remaining roots are all of the form

$$aA + b\sqrt[n]{K'} + c\sqrt[n]{K''} + d\sqrt[n]{K'''} + \ldots + l\sqrt[n]{K^{(n-1)}}$$

in which $a, b, c, d, \ldots\ldots l$ denote certain functions of α. But since these roots must always remain the same, whatever primitive root we substitute for α, $\therefore$ amongst the remaining roots $x', x'', x''', \ldots\ldots x^{(n-1)}$ there must at least be another, which has the form of the root $x^{(n)}$; and since this is not possible, α must quite vanish from the value of $x^{(n)}$ already found, and $\therefore \alpha^{\nu+1} = \alpha n = 1$, consequently $\nu = n - 1$. Therefore

$$x = \frac{A}{n} + \frac{1}{n}(\sqrt[n]{K'} + \sqrt[n]{K''} + \sqrt[n]{K'''} + \ldots + \sqrt[n]{K^{(n-1)}})$$

is a root of the given equation, and the remaining ones may be determined from the equations in 13, when we substitute in them $n - 1$ for ν.

Remark. The solution which I have here given, has only this one fault, that we do not by its means obtain all the roots at once, but only one, and that the remaining ones must afterwards be sought by a very troublesome elimination. I shall $\therefore$ give another solution, which has not this fault, and in other respects also is perhaps preferable to the former one. For the sake of perspicuity I shall begin with an equation of the fifth degree.

SECTION CLXVII.

PROB. Solve the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

so as to obtain all the roots at once.

Solution 1. As in the preceding §, let $t^5 - K = 0$ be the equation for the recurring period of all the roots of the function

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \alpha^4 x^v$$

then

$$K = t^5 = (x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \alpha^4 x^v)^5$$

or also

$$K = (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{\prime v} + x^v)^5$$

and K can have no other unequal values of forms, as we have already seen, but those which arise from the 24 transpositions of the four first roots.

2. Thus far all is the same, as in the foregoing solution. But further, instead of forming the equation

$$K^4 - pK^3 + qK^2 - rK + s = 0$$

as heretofore, from the recurring period of the four first roots, I shall now assume that it has the four following values of forms for roots:

$$(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{\prime v} + x^v)^5$$
$$(\alpha^2 x' + \alpha^4 x'' + \alpha x''' + \alpha^3 x^{\prime v} + x^v)^5$$
$$(\alpha^3 x' + \alpha x'' + \alpha^4 x''' + \alpha^2 x^{\prime v} + x^v)^5$$
$$(\alpha^4 x' + \alpha^3 x'' + \alpha^2 x''' + \alpha x^{\prime v} + x^v)^5$$

the three last of which are obtained from the first one, when in it we substitute successively α^2, α^3, α^4 for α.

3. In these four values of K, x^{IV} is always combined with another power of α, and $\therefore$ all the 24 unequal values of K may be derived from these, merely by permuting the roots x', x'', x'''. Thus, if the roots x^{IV}, x^{V}, retain their places, and we merely permute the three first roots, then each of the above four values of forms gives five new ones, and consequently all together give the 24 values of K.

4. Since the coefficients p, q, r, s, are symmetrical functions of the four above-mentioned values of K, $\therefore$ these functions undergo no change, either by the recurring transposition of all the roots, or by the substitution of α for α^2, α^3, α^4; and consequently they can have no more unequal values than those which arise exclusively from the transposition of the three first roots x', x'', x'''. Therefore these functions depend on equations of the sixth degree only.

5. Consequently, if we put $p = f: (12345)$, then $f: (12345)$, $f: (23145)$, $f: (31245)$, $f: (21345)$, $f: (13245)$, $f: (32145)$, are the six values of forms of p. Now if we assume that the three values of forms $f: (12345)$, $f: (23145)$, $f: (31245)$, which arise from the recurring transposition of the three first roots, are given by the equation

$$p^3 - p'p^2 + p'p - r' = 0,$$

then the coefficients p', q', r', are such functions of x', x'', x''', x^{IV}, x^{V}, as can only have the single value, which the substitution of x' for x'' gives. Therefore p' (and the

same obtains of q' and r') has no more unequal values of forms than the two f' : (12345), f' : (21345). Therefore p' depends on an equation of the second degree only.

6. Let

$$p'^2 - p''p' + q'' = 0$$

be this equation; then p'', q'', are symmetrical functions of x', x'', x''', x^{IV}, x^{V}, and consequently may be expressed rationally by the coefficients A, B, C, D, E of the given equation.

7. We now have the three following equations:

$$\begin{aligned} &K^4 - qK^3 + qK^2 - rK + s = 0 \\ &p^3 - p'p^2 + q'p - r' = 0 \\ &p'^2 - p''p' + r'' = 0. \end{aligned}$$

The last gives the value of p', from which, by the foregoing chapter, the coefficients q', r' may be determined. The solution of the second equation again will give the coefficient p; and from this again we may find the coefficients q, r, s. Now, by solving the first equation, we obtain four values of K.

8. Let K', K'' K''', K^{IV}, be these four values, we then have the four following equations (2):

$$\begin{aligned} (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + x^{V})^5 &= K' \\ (\alpha^2 x' + \alpha^4 x'' + \alpha x''' + \alpha^3 x^{IV} + x^{V})^5 &= K'' \\ (\alpha^3 x' + \alpha x'' + \alpha^4 x''' + \alpha^2 x^{IV} + x^{V})^5 &= K''' \\ (\alpha^4 x' + \alpha^3 x'' + \alpha^2 x''' + \alpha x^{IV} + x^{V})^5 &= K^{IV}. \end{aligned}$$

If we extract the fifth root from both parts of these

equations, we then have, including the equation $x' + x'' + x''' + x^{IV} + x^{V} = A$,

$$\begin{aligned}
x' + x'' + x''' + x^{IV} + x^{V} &= A \\
\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + x^{V} &= \sqrt[5]{K'} \\
\alpha^2 x' + \alpha^4 x'' + \alpha x''' + \alpha^3 x^{IV} + x^{V} &= \sqrt[5]{K''} \\
\alpha^3 x' + \alpha x'' + \alpha^4 x''' + \alpha^2 x^{IV} + x^{V} &= \sqrt[5]{K'''} \\
\alpha^4 x' + \alpha^3 x'' + \alpha^2 x''' + \alpha x^{IV} + x^{V} &= \sqrt[5]{K^{IV}}.
\end{aligned}$$

9. If we add these equations together, we then obtain, since $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = [1] = 0$

$$5x^{V} = A + \sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{IV}}.$$

If we multiply the second by α^4, the third by α^3, the fourth by α^2, the fifth by α, and then add them to the first, we obtain

$$5x' = A + \alpha^4\sqrt[5]{K'} + \alpha^3\sqrt[5]{K''} + \alpha^2\sqrt[5]{K'''} + \alpha\sqrt[5]{K^{IV}}.$$

If we multiply the second by α^3, the third by α, the fourth by α^4, the fifth by α^2, and then add them to the first, we obtain

$$5x'' = A + \alpha^3\sqrt[5]{K'} + \alpha\sqrt[5]{K''} + \alpha^4\sqrt[5]{K'''} + \alpha^2\sqrt[5]{K^{IV}}.$$

If we multiply the second by α^2, the third by α^4, the fourth by α, the fifth by α^3, and then add them to the first, we obtain

$$5x''' = A + \alpha^2\sqrt[5]{K'} + \alpha^4\sqrt[5]{K''} + \alpha\sqrt[5]{K'''} + \alpha^3\sqrt[5]{K^{IV}}.$$

Lastly, if we multiply the second by α, the third by α^2, the fourth by α^3, the fifth by α^4, and then add them to the first, we obtain

$$5x^{IV} = A + \alpha\sqrt[5]{K'} + \alpha^2\sqrt[5]{K''} + \alpha^3\sqrt[5]{K'''} + \alpha^4\sqrt[5]{K^{IV}}.$$

10. If we inspect the values of the roots x', x'', x''', x^{IV}, x^{V}, as they have been here found, we shall immediately observe, that if in any one of the four last, we substitute successively α^2, α^3, α^4, α^5, for α, we always obtain the other four. Therefore, all the roots of the given equation may be comprehended in the following expression:

$$x = \frac{1}{5}(A + \alpha\sqrt[5]{K'} + \alpha^2\sqrt[5]{K''} + \alpha^3\sqrt[5]{K'''} + \alpha^4\sqrt[5]{K^{IV}})$$

when by α we suppose each root of the equation $x^5 - 1$.

11. I shall only further remark, that in this solution the root α must certainly vanish in the coefficients p, q, r, s. For, since the functions K', K'', K''', K^{IV}, in 8 are such, that in the substitution of α for α^2, α^3, α^4, consequently for the remaining roots β, γ, δ, of the equation $x^5 - 1 = 0$, they merely change places; $\therefore$ the coefficients p, q, r, s, as symmetrical functions of K', K'', K''', K^{IV}, must also remain the same in thiss subtitution, and consequently must be symmetrical functions of α, β, γ, δ; $\therefore$, by the fifth chapter, are rational.

SECTION CLXVIII.

PROB. Solve the general equation of the undetermined nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

in such a way, that all the roots may be obtained at once, and yet under the supposition that n is a prime number.

Solution 1. As in § CLXVI, let

$$t = x' + \alpha x'' + \alpha^2 x''' \ldots\ldots + \alpha^{n-1} x^{(n)}$$

$\therefore$

$$K = t^n = (x' + \alpha x'' + \alpha^2 x''' + \ldots\ldots + \alpha^{n-1} x^{(n)})^n$$

or, which is the same,

$$K = (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \ldots + \alpha^{n-1} x^{(n-1)} + x^{(n)})^n.$$

The function K, as we have already seen in the above-named §, is then such, that it remains the same in the recurring transpositions of all the roots x', x'', x''',...$x^{(n)}$, and consequently can have no more unequal values than those which arise from the transposition of the $n-1$ first roots.

2. In order to find the $1.2.3\ldots n-1$ values of forms of the function K, which arise from the transposition of the $n-1$ first roots, we first substitute α^4, α^3, α^2,...α^{n-1} for α; hence there arise the following $n-1$ values:

$$(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \ldots\ldots + \alpha^{n-1} x^{(n-1)} + x^{(n)})^n$$
$$(\alpha^2 x' + \alpha^4 x'' + \alpha^6 x''' + \ldots\ldots + \alpha^{n-2} x^{(n-1)} + x^{(n)})^n$$
$$(\alpha^3 x' + \alpha^6 x'' + \alpha^9 x''' + \ldots\ldots + \alpha^{n-3} x^{(n-1)} + x^{(n)})^n$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$(\alpha^{n-1} x' + \alpha^{n-2} x'' + \alpha^{n-3} x''' + \ldots\ldots + \alpha x^{(n-1)} + x^{(n)})^n.$$

Since in all these values of forms, because n is a prime number, the same powers of α occur, and in each the root $x^{(n-1)}$ is combined with another power of α; it is $\therefore$ evident, that we obtain all the values of K, when in these $n-1$ values, we permute the roots x', x'', x''',... ...$x^{(n-2)}$ in all possible ways, but let the root $x^{(n-1)}$ retain its place. Each of these values then gives (including the one under consideration) $1 . 2 . 3\ldots\ldots n-2$

values, and consequently all together give all the above $1 \cdot 2 \cdot 3 \ldots n-1$ values of K.

3. Now if we assume that the $n-1$ values in 2 are the roots of the equation

$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$

consequently, by the fifth chapter, the root α must vanish in the coefficient s p, q, r, &c. and they ∴ remain the same when α is substituted for α^2, α^3, α^4, .. α^{n-1}. Hence, however, it necessarily follows, that these coefficients can have no more unequal values than those which arise from the transposition of the $n-2$ roots x', x'', x''',...$x^{(n-1)}$.

4. Since ∴ the coefficients p, q, r, &c. have no more unequal values than those which arise from the transposition of the $n-2$ first roots; consequently these are similarly circumstanced with the coefficients mentioned in § CLXVI. Thus the equation of the $1 \cdot 2 \cdot 3 \ldots$ $\ldots n-2$th degree, on which the coefficient p depends, may, by the union of those of its values which arise from the recurring transposition of the $n-2$ first roots, be reduced to an equation

$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} \&c. = 0$$

whose coefficients p', q', r', &c. only depend now on equations of the $1 \cdot 2 \cdot 3 \ldots n-3$th degree. Further, the equation for p', by uniting its values of forms, which arise from the recurring transposition of the $n-3$ first roots, may be reduced to an equation

$$p'^{n-3} - p''p'^{n-4} + q''p'^{n-5} - r''p'^{n-6} + \&c. = 0$$

whose coefficients p'', q'', r'', &c. only depend on equa-

tions of the $1 . 2 . 3 \ldots n-4$th degree, and so on, till we come to an equation of the second degree.

5. Having determined the values of the coefficients p, q, r, &c. by the solution of all these equations successively, and by means of the foregoing chapter, then the solution of the equation $K^{n-1} - pK^{n-2} + \&c. = o$, gives $n-1$ values of K, which I shall denote by K', K'', $K''', \ldots\ldots K^{(n-1)}$. The $n-2$ values in R correspond to these values; we $\therefore$ have the following $n-1$ equations:

$$(\alpha x' + \alpha^2 x'' + \ldots\ldots + \alpha^{n-1} x^{(n-1)} + x^{(n)})^n = K'$$
$$(\alpha^2 x' + \alpha^4 x'' + \ldots\ldots + \alpha^{n-1} x^{(n-1)} + x^{(n)})^n = K''$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(\alpha^{n-1} x' + \alpha^{n-2} x'' + \ldots + \alpha x^{(n-1)} + x^{(n)})^n = K^{(n-1)}.$$

6. By extracting the nth root there arise the following equations:

$$x' + x'' + x''' + \ldots\ldots + x^{(n)} = A$$
$$\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K'}$$
$$\alpha^2 x' + \alpha^4 x'' + \alpha^6 x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K''}$$
$$\alpha^3 x' + \alpha^6 x'' + \alpha^9 x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K'''}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha^{n-1} x' + \alpha^{n-2} x'' + \alpha^{n-3} x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K^{(n-1)}}.$$

If we multiply the second equation by α^{n-1}, the third by α^{n-2}, the fourth by α^{n-4}, and so on, and then add them to the first, we get

$$nx' = A + \alpha^{n-1} \sqrt{K'} + \alpha^{n-2} \sqrt{K''} + \ldots\ldots + \alpha \sqrt{K^{(n-1)}}.$$

If we multiply the second by α^{n-2}, the third by α^{n-4}, the

fourth by α^{n-6}, and so on, and then add them to the first, we get

$$nx'' = A + \alpha^{n-2}\sqrt[n]{K'} + \alpha^{n-4}\sqrt[n]{K''} + \ldots\ldots + \alpha^{2}\sqrt[n]{K^{(n-1)}}.$$

In a similar way, we further find

$$nx''' = A + \alpha^{n-3}\sqrt[n]{K'} + \alpha^{n-6}\sqrt[n]{K''} + \ldots\ldots + \alpha^{3}\sqrt[n]{K^{(n-1)}}$$

$$nx^{lv} = A + \alpha^{n-4}\sqrt[n]{K'} + \alpha^{n-8}K'' + \ldots\ldots + \alpha\sqrt[n]{K^{(n-1)}}$$

&c.

Lastly, if we add all the n equations together, we get

$$nx^{(n)} = A + \sqrt[n]{K'} + \sqrt[n]{K''} + \ldots + \sqrt[n]{K^{(n-1)}}.$$

It is readily seen, that we can derive the values of x'', x''', x^{lv},......$x^{(n)}$ from the value of x', by substituting in it α^2, α^3, α^4,......α^n $(=1)$ successively for α, consequently by substituting for α all the roots of the equation $x^n - 1 = 0$. We $\therefore$ have the following general expression for the roots of the given equation:

$$x = \frac{1}{n}(A + \alpha\sqrt[n]{K'} + \alpha^{2}\sqrt[n]{K''} + \ldots\ldots + \alpha^{n-1}\sqrt[n]{K^{(n-1)}}.$$

REMARK. When n is a compound number, then indeed, for the reasons given in 1, § CXXXVII, this method is not applicable; but for this case, I shall in the Third Part give a particular method, which has this advantage besides, that it is much shorter.

T. C. HANSARD, Paternoster-Row Press.

ERRATA.

The author has in the original designated the sum of the roots of an equation, the sum of their squares, cubes, and in general the sum of their μth powers by the symbols [1], [2], [3]....[μ], and has employed the *crotchet* for this particular purpose, instead of the *parenthesis*. By a mistake in the printing, which was not discovered till it was too late to be corrected, the parenthesis has been used indiscriminately for this and the usual purposes through the earlier part of the volume. This gives rise to a little confusion, the mere mention of which will be sufficient to prevent any obscurity which might otherwise have arisen. See, in particular, pages 5, 6, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 35, 36, 58, 66, 67, 69, 82, 83, 84, 85, 86, 87, 133, 134, where it will be most material to attend to this distinction.

Page 5, *line* 6, *for* function, *read* symmetrical function.

—— 11, —— 3, 4, *for* of the *m*th order, *read* taken *m* and *m*.

—— 13, —— 15, *for* letters, *read* roots.

——, —— 18 - - - ditto.

—— 21, —— 6, *for* $=(\beta-\alpha)$ *read* $-(\beta-\alpha)$.

—— 22, —— 23, *for* evolution, *read* development.

—— 41, —— 6, 7, *for* odd or even, *read* even or odd.

—— 49, —— 16, *for* number, *read* number of divisions.

——, —— 20, *for* numbers of a different kind, *read* numbers of different kinds of things.

—— 57, —— 18, The product in this line should be written thus: $(\mp m)^{\mu'}\times(\mp m')^{\mu'}\times(\mp m'')^{\mu''}$.

Page 59, *line* 9, 10, *for* in each transformation are changed, *read* when transformed are unchanged.

—— 61, —— 11, *for* the, *read* this.

—— 68, —— 14, *for* t^3, *read* x^3.

—— 99, —— 13, *for* B_α, *read* B_κ.

—— 106, —— 8, 9, *for* § 41 and 42, *read* § 40 and 41.

—— 109, —— 1, 2, *for* when the roots are transformed and permuted, *read* by the substitution and permutation of the roots.

—— 113, —— 1, *dele* them.

—— 130, —— 23, *for* quite as, *read* as little.

—— 132, —— 6, *for* other, *read* others.

—— 133, —— 5, The coefficient of the second term should be x'.

—— 135, —— 12, 13, *for* transformation, *read* substitution.

—— 143, —— 23, *for* merely, *read* all.

—— 144, —— 4, *for* other, *read* higher.

—— 149, —— 3, The letters A', B', C', D', ought to have been old English capitals.

—— 152, —— 3, 4, The letters A', B', C', D', E', F' should have been old English capitals.

—— 209, —— 10, *for* referred, *read* in reference.

—— 226, —— 4, *dele* try to.

TABLE I.

2

	A^2	B
[2] =	1	− 2
$[1^2]$ =		1

5

	A^5	A^3B	AB^2	A^2C	BC	AD	E
	1	− 5	+ 5	+ 5	− 5	− 5	+ 5
=		1	− 3	− 1	+ 5	+ 1	− 5
[23] =			1	− 2	− 1	+ 5	− 5
$[1^23]$ =				1	− 2	− 1	+ 5
$[12^2]$ =					1	− 3	+ 5
$[1^32]$ =						1	− 5
$[1^5]$ =							1

	A^6	A^4B	A^2B^2	B^3	A^3D	ABD	CD	A^2E	BE	AF	G
[6] =	1	− 6	+ 9	− 2	- 7	+14	− 7	+ 7	− 7	− 7	+ 7
[13] =		1	− 4	+ 2	- 1	− 8	+ 7	− 1	+ 7	+ 1	− 7
[24] =			1	− 2	- 2	− 4	+ 7	− 2	− 3	+ 7	− 7
$[3^2]$ =				1	- 3	− 2	− 5	− 7	+ 7	+ 7	− 7
$[1^24]$ =					- 1	+ 3	− 7	+ 1	− 2	− 1	+ 7
[123					- 3	+ 8	− 2	+ 3	− 4	− 8	+14
					*	− 1	+ 5	+ 4	− 7	− 4	+ 7
					*	− 2	− 1	+ 2	+ 3	− 7	+ 7
					1	− 3	+ 3	− 1	+ 2	+ 1	− 7
] =						1	− 3	− 4	+ 6	+ 9	−21
$[12^3]$ =							1	*	− 3	+ 5	− 7
$[1^43]$ =								1	− 2	− 1	+ 7
$[1^32^2]$ =									1	− 5	+ 1
$[1^52]$ =										1	− 7
$[1^7]$ =											1

TABLE II.

	A^8	A^6B	A^4B^2	A^2B^3	B^4	A^5C	A^3BC	AB^2C	A^2C^2	BC^2	A^4D	A^2BD	B^2D	ACD	D^2	A^3E	ABE	CE	A^2F	BF	AG	H
[8] =	1	− 8	+20	−16	+ 2	+ 8	−32	+24	+12	− 8	− 8	+24	− 8	−16	+ 4	+ 8	−16	+ 8	− 8	+ 8	+ 8	− 8
[17] =		1	− 6	+ 9	− 2	− 1	+11	−17	− 5	+ 8	+ 1	−10	+ 8	+ 9	− 4	− 1	+ 9	− 8	+ 1	− 8	− 1	+ 8
[26] =			1	− 4	+ 2	− 2	+ 8	*	− 9	+ 2	+ 2	− 6	− 4	+16	− 4	− 2	+ 4	− 8	+ 2	+ 4	− 8	+ 8
[35] =				1	− 2	*	− 3	+ 6	+ 3	− 7	+ 3	− 9	+ 8	+ 1	− 4	− 3	+ 1	+ 7	+ 8	− 8	− 8	+ 8
[4^2] =					1	*	*	− 4	+ 2	+ 4	*	+ 4	− 4	− 8	+ 6	− 4	+ 8	− 4	+ 4	− 4	− 4	+ 4
[1^26] =						1	− 5	+ 5	+ 5	− 5	− 1	+ 4	− 2	− 9	+ 4	+ 1	− 3	+ 8	− 1	+ 2	+ 1	− 8
[125] =							1	− 3	− 1	+ 5	− 3	+11	− 4	−10	+ 8	+ 3	− 8	+ 1	− 3	+ 4	+ 9	−16
[134] =								1	− 2	− 1	*	− 1	*	+10	− 8	+ 4	−10	+ 1	− 9	+16	+ 9	−16
[2^24] =									1	− 2	*	− 2	+ 4	*	− 4	+ 2	− 4	+ 8	− 2	− 4	+ 8	− 8
[23^2] =										1	*	*	− 2	− 1	+ 4	*	+ 5	− 7	− 5	+ 2	+ 8	− 8
[1^35] =											1	− 4	+ 2	+ 4	− 4	− 1	+ 3	− 3	+ 1	− 2	− 1	+ 8
[1^224] =												1	− 2	− 1	+ 4	− 4	+11	− 9	+ 4	− 6	−10	+24
[1^23^2] =													1	− 2	+ 2	*	− 1	+ 3	+ 5	− 9	− 5	+12
[12^23] =														1	− 4	*	− 3	+ 6	+ 5	*	−17	+24
[2^4] =															1	*	*	− 2	*	+ 2	− 2	+ 2
[1^44] =																1	− 3	+ 3	− 1	+ 2	+ 1	− 8
[1^323] =																	1	− 3	− 5	+ 8	+11	−32
[1^22^3] =																		1	*	− 4	+ 9	−16
[1^53] =																			1	− 2	− 1	+ 8
[1^42^2] =																				1	− 6	+20
[1^62] =																					1	− 8
[1^8] =																						1

TABLE III.

	BCE	ADE	E^2	A^4F	A^2BF	B^2F	ACF	DF	A^3G	ABG	CG	A^2H	BH	AI	K
)	−20	−20	+ 5	−10	+30	−10	−20	+10	+10	−20	+10	−10	+10	+10	−10
2	+20	+11	− 5	+ 1	−12	+10	+11	−10	− 1	+11	−10	+ 1	−10	− 1	+10
2	+ 4	+20	− 5	+ 2	− 6	− 6	+20	−10	− 2	+ 4	−10	+ 2	+ 6	−10	+10
9	− 1	+40	− 5	+ 3	− 9	+10	− 1	−10	− 3	− 1	+11	+10	−10	−10	+10
6	+20	−44	− 5	+ 4	− 6	− 2	− 4	+14	−10	+20	−10	+10	−10	−10	+10
0	−15	−15	+10	+ 5	−15	+ 5	+10	− 5	− 5	+10	− 5	+ 5	− 5	− 5	+ 5
2	−12	−11	+ 5	− 1	+ 4	− 2	−11	+10	+ 1	− 3	+10	− 1	+ 2	+ 1	−10
3	− 3	−31	+10	− 3	+11	− 4	−10	+20	+ 3	− 8	− 1	− 3	+ 4	+11	−20
5	−19	− 7	+10	− 4	+15	− 8	− 4	− 4	+ 4	−10	− 1	−11	+20	+11	−20
[illegible]	+10	+23	−15	− 5	+18	− 8	− 7	− 4	+11	−31	+20	−11	+20	+11	−20
0	− 4	− 8	+ 5	− 2	+ 6	*	− 8	− 2	+ 2	− 4	+10	− 2	− 6	+10	−10
4	+17	+10	−15	− 5	+15	− 4	−19	+20	+ 5	− 3	− 1	−12	+ 4	+20	−20
2	−12	+ 4	+ 5	*	− 6	+10	+ 4	−14	+ 6	−12	+10	− 6	+ 2	+10	−10
3	+ 1	− 8	+ 5	*	− 3	− 4	+13	− 2	+ 3	+ 1	−11	−10	+10	+10	−10
5	+ 5	+11	− 5	+ 1	− 4	+ 2	+ 4	−10	− 1	+ 3	− 3	+ 1	− 2	− 1	+10
9	+15	+18	−15	+ 4	−15	+ 6	+15	− 6	− 4	+11	− 9	+ 4	− 6	−12	+30
	− 4	−12	+10	+ 5	−19	+10	+15	− 6	− 5	+13	− 9	+12	−22	−12	+30
2	+ 1	− 8	+ 5	*	+ 1	− 1	− 3	+ 9	− 6	+17	−15	+ 6	−11	− 6	+15
2	−13	− 1	+10	+ 5	−17	+ 4	+18	−18	− 5	+12	− 9	+ 5	+ 2	−21	+30
4	+ 5	*	− 5	*	+ 7	− 8	−15	+12	−12	+21	+ 3	+26	−28	−42	+60
*	− 1	+ 5	− 5	*	*	+ 4	− 7	+ 2	*	− 4	+11	+ 7	−10	− 7	+10
2	+ 4	*	− 5	*	+ 2	− 4	*	+10	− 2	+ 4	−10	+ 2	+ 6	−10	+10
*	− 2	− 1	+ 5	*	*	+ 2	+ 3	− 9	*	− 7	+ 6	+ 7	+ 1	−15	+15
5	− 5	− 5	+ 5	− 1	+ 4	− 2	− 4	+ 4	+ 1	− 3	+ 3	− 1	+ 2	+ 1	−10
1	+ 5	+ 1	− 5	− 5	+19	− 8	−19	+16	+ 5	−14	+12	− 5	+ 8	+13	−40
2	− 1	+ 5	− 5	*	− 1	*	+ 5	− 8	+ 6	−16	+12	−13	+24	+13	−40
1	− 2	− 1	+ 5	*	− 4	+ 8	+ 3	−12	+ 9	−23	+18	− 9	+ 4	+33	−60
=	1	− 3	+ 5	*	*	− 4	+ 6	*	*	+ 9	−24	−21	+28	+33	−60
	$2^{3}3]$ =	1	− 5	*	*	*	− 3	+ 8	*	+ 5	− 2	− 7	− 8	+31	−40
		$[2^{5}]$ =	1	*	*	*	*	− 2	*	*	+ 2	*	− 2	+ 2	− 2
			$[1^{5}5]$ =	1	− 4	+ 2	+ 4	− 4	− 1	+ 3	− 3	+ 1	− 2	− 1	+10
				$[1^{4}24]$ =	1	− 2	− 1	+ 4	− 6	+17	−15	+ 6	−10	−14	+50
					$[1^{4}3^{2}]$ =	1	− 2	+ 2	*	− 1	+ 3	+ 7	−13	− 7	+25
						$[1^{3}2^{2}3]$ =	1	− 4	*	− 5	+12	+14	−12	−46	+100
							$[1^{2}2^{4}]$ =	1	*	*	− 4	*	+ 9	−16	+25
								$[1^{6}4]$ =	1	− 3	+ 3	− 1	+ 2	+ 1	−10
									$[1^{5}23]$ =	1	− 3	− 7	+12	+15	−60
										$[1^{4}2^{3}]$ =	1	*	− 6	+20	−50
											$[1^{7}3]$ =	1	− 2	− 1	+10
												$[1^{6}2^{2}]$ =	1	− 8	+35
													$[1^{8}2]$ =	1	−10
														$[1^{10}]$ =	1